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THE MODULI SPACE OF BRANCHED SUPERMINIMAL SURFACES OF A FIXED DEGREE, GENUS AND CONFORMAL STRUCTURE IN THE FOUR-SPHERE

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0. Introduction

Minimal immersions from a Riemann surface M into S^n were studied by Calabi ([3]) and Chern ([4]), among many authors. To each such immersion F in S^4, they found a holomorphic quartic form Q_F (to be defined in Section 1) on M. A superminimal immersion is one for which $Q_F=0$, which is always the case when $M=S^2$. In [2], Bryant studied a superminimal immersion of a higher genus into S^4 by lifting it to $\boldsymbol{CP}^3$, the twistor space of S^4. The lift of a superminimal immersion is a holomorphic curve, of the same degree as that of the immersion, which is horizontal with respect to the twistorial fibration; more precisely, it is a holomorphic curve in $\boldsymbol{CP}^3$ satisfying the differential equation $z_0dz_1-z_1dz_0+z_2dz_3-z_3dz_2=0$. Setting $z_0=1$, $z_1+z_2z_3=f$ and $z_2=g$, one can solve z_1, z_2, z_3 in terms of the meromorphic functions f and g, which serves as a kind of "Weierstrass representation". Via this representation, Bryant showed the existence of a superminimal immersion from any compact Riemann surface into S^4. However, his existence result does not specify the degree d of the immersion, which is the simplest global invariant of the surface.

In Loo ([12]) and Verdier ([17]), $f_1=z_1/z_0$ and $f_2=z_3/z_2$ were chosen in place of the aforementioned f and g. Generically, f_1 and f_2 are of degree d which satisfy $\mathrm{ram}(f_1)=\mathrm{ram}(f_2)$, where $\mathrm{ram}(f)$ denotes the ramification divisor of the meromorphic function f. This gives a scheme of constructing the moduli space of all branched superminimal surfaces in S^4 with a fixed degree d. For $M=S^2$, Loo ([12]) showed that the moduli space is connected and has dimension $2d+4$; Verdier ([17]) in addition pointed out that the moduli space has three irreducible components.

In this paper, we propose to carry the investigation over to higher genera. Let $F: M\to S^4$ be a superminimal immersion of degree d and let $\tilde{F}: M\to \boldsymbol{CP}^3$ be its horizontal lift. Let L_F be the pullback bundle via $\tilde{F}$ of the hyperplane bundle of $\boldsymbol{CP}^3$. We may regard $z_0,\cdots,z_3$ as four sections in $H^0(L_F)$ without common zeros. Now there is a natrual map $\mathscr{Ram}$ which sends the 1-dimensional linear system $\langle z_0,z_1\rangle$ (called a g_d^1), i.e., the plane spanned by z_0, z_1 in the Grassmann manifold $G(2,H^0(L))$ of two-planes is $H^0(L_F)$, to the zero divisor of $z_0dz_1-z_1dz_0$