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A MATHEMATICAL JUSTIFICATION FOR KORTEWEG-DE VRIES EQUATION AND BOUSSINESQ EQUATION OF WATER SURFACE WAVES

Dedicated to Professor Sigeru Mizohata on the occasion of his sixtieth birthday

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1 Introduction

1. In this paper, we give a mathematically rigorous derivation of Korteweg—de Vries equation[1] and of Boussinesq equation from the Euler equation for surface wave of water in irrotational motion.

In deriving these equations, we begin with noticing the following two facts: (i) we have solved in [6] the Cauchy problem for two space dimensional water surface wave equations in a class of analytic functions locally in time, in the dimensionless form, (ii) the surface wave and the velocity potential depend on the dimensionless parameter δ introduced in this dimensionless problem as the ratio of the water depth to wave length in such a way that they are infinitely many times differentiable with respect to δ [6].

We can expand[2] equations with respect to δ^2. In dropping the terms of order $O(\delta^4)$ in this expansion we get K-dV equation and "Boussinesq equation" and we prove that their solutions approximate the original water surface wave up to the same order of errors as the dropped terms with respect to δ.

Our "Boussinesq equation" has not the same form as the original equation given by Boussinesq himself in 1871 [3]. If one substitutes the "first approximation" $\varphi_t = \varphi_x + O(\delta^2)$ in the terms of order $O(\delta^2)$ of our "Boussinesq equation", one can immediately recover the original one. However this substitution is not justified generally. We remark later that this substitution would be rather destructive for good approximation even for waves for which it is justified, see §2. In the last part of the paragraph 2, we compare our derivation of "Boussinesq" equation with the original study of Boussinesq.

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1) Mentioned simply K-dV equation hereafter.

2) This is not Friedrichs expansion [7]. The precise meaning of this expansion is given in §2.