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THE VANISHING OF COHOMOLOGY ASSOCIATED TO DISCRETE SUBGROUPS OF COMPLEX SIMPLE LIE GROUPS*

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1. Introduction

Let G denote a connected complex simple Lie group and K a maximal compact subgroup of G. The quotient $M=G/K$ is a riemannian symmetric space of non-compact type. Let Γ denote a discrete subgroup of G with compact quotient $\Gamma\backslash G$, and let ρ denote an irreducible non-trivial complex representation of G in a finite dimensional complex vector space F. In this paper we prove that for such representations a certain quadratic form defined by Matsushima and Murakami [3] is positive definite, and hence $H^*(\Gamma, M, \rho)$ vanishes.

The motivation for this paper is a result of Min-Oo and Ruh [4] on comparison theorems for non-compact symmetric spaces, where an estimate from below for the first eigenvalue of the Laplace operator on 2-forms with values in a bundle associated to the adjoint representation is essential. This estimate is an immediate consequence of the positivity of the above quadratic form. The vanishing of $H^*(\Gamma, M, \rho)$, without the information on the first eigenvalue, is a special case of [1, Ch. VII, Th. 6. 7].

2. The result

Let $\mathfrak{g}$ denote the Lie algebra of left-invariant vector fields of the simple Lie group G, $\rho\colon \mathfrak{g}\to\mathfrak{gl}(F)$ the representation induced by $\rho\colon G\to GL(F)$, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ with $\mathfrak{k}$ the Lie algebra of a maximal compact subgroup K. We identify the Lie algebra $\mathfrak{g}$ with the corresponding vector fields on $\Gamma\backslash G$.

Let $A(\Gamma, M, \rho)$ ($A_0(\Gamma, M, \rho)$ in the notation of Matsushima and Murakami [3]) denote the vector space of F-valued differential forms on $\Gamma\backslash G$ which are horizontal and ad$K$-equivariant, i.e., $\eta\in A(\Gamma, M, \rho)$ satisfies $i_X\eta=0$ and $\theta_X\eta=-\rho(X)\eta$ for all $X\in\mathfrak{k}$, where i_X is interior multiplication and θ_X is the Lie

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