Kanzaki, T.
Osaka J. Math.
10 (1973), 607–609

A NOTE ON THE ORTHOGONAL GROUP OF A QUADRATIC MODULE OF RANK TWO OVER A COMMUTATIVE RING

TERUO KANZAKI

(Received April 24, 1973)

Let A be an arbitrary commutaive ring with the identity element. This note will give an elementary property on the orthogonal group of a non-degenerate quadartic A-module of rank two. Throughout this paper, we will assume that (V, q) is a non-degenerate quadratic A-module such that V is a finitely generated projective A-module and $[V_{\mathfrak{m}}: A_{\mathfrak{m}}]=2$ for all maximal ideal $\mathfrak{m}$ of A. The Cilfford algebra $\mathrm{C}(V, q)$ is a quadratic extension of $\mathrm{C}_0(V, q)$, the set of homogeneous elements of degree 0 in $\mathrm{C}(V, q)$, and $\mathrm{C}_0(V, q)$ is a commutative and separable quadratic extension of A (cf. [3], [4]). Set $B=\mathrm{C}_0(V, q)$. B is a Galois extension of A with a Galois group $G=\{I, \tau\}$, and τ is the unique A-algrbra automorphism of B such that the fixed subring of B is A ([4], [5]). By [3], V is an invertible B-bilmodule, and (V, ϕ), $\phi\colon V\times V\to B$; $\phi(x, y)=xy$ in $\mathrm{C}(V, q)$ for $x, y\in V$, is a non-degenerate hermitian B-module ((2.4) in [3]). We denote by $\mathrm{I}(A)$ the set of idempotents in A, which is an abelian group with respect to the product $*$; $e*e'=e+e'-2ee'$ for $e, e'\in \mathrm{I}(A)$. Then, by [1], the group Aut (B/A) of all A-algebra automorphisms of B is $\{e\tau+(1-e)I;\, e\in I(A)\}$, and is isomorphic to $\mathrm{I}(A)$ by the isomorphism $\mu\colon \mathrm{I}(A)\to \mathrm{Aut}\ (B/A)$; $e \rightsquigarrow \mu =e\tau+(1-e)I$. Let $O(V, q)$ be the orthogonal group of (V, q), i.e. $O(V, q)=\{\rho\in \mathrm{Hom}_A(V, V);\, q(\rho v))=q(v)$ and $\rho(V)=V\}$. For any $\rho\in O(V, q)$, ρ is extended to an A-algebra automorphism $\tilde{\rho}$ of $\mathrm{C}(V, q)$ which induces an automorphism of B. Accordingly, there exists a group homorphism $\eta\colon O(V, q)\to \mathrm{I}(A)$; $\rho \rightsquigarrow \mu^{-1}(\rho\,|\,B)$. We put $O^+(V, q)=\{\rho\in O(V, q);\, \rho\,|\,B=I\}$ and $O^-(V, q)=\{\rho\in O(V, q);\, \tilde{\rho}\,|\,B\neq I\}$.

REMARK 1. Let V be a free A-module with the basis $\{u, v\}$, $V=Au\oplus Av$. For $\rho\in O(V, q)$, let $\rho=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ denote the matrix of ρ with respect to the basis $\{u, v\}$. Then $(\det \rho)^2=1$. If ρ is in $O^+(V, q)$ then $\det \rho=1$. If $\tilde{\rho}\,|\,B=\tau$ then $\det \rho=-1$.

Proof. Since $\mathrm{C}(V, q)=A\oplus Auv\oplus Au\oplus Av$ and $B=A\oplus Auv$, we have $\tilde{\rho}(uv)=(au+bv)(cu+dv)=\mathrm{B}_q(cu, bv)+acq(u)+bdq(v)+(\det \rho)uv$. Since $(uv)^2=\mathrm{B}_q(u, v)uv-q(u)q(v)$, we have $B=\mathrm{C}^+(V, q)=(A, \mathrm{B}_q(u, v), -1)$ and $\tau(uv)=\mathrm{B}_q(u, v)$