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MULTIPLICATIVE P-SUBGROUPS OF SIMPLE ALGEBRAS

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Amitsur ([1]) determined all finite multiplicative subgroups of division algebras. We will try to determine, more generally, multiplicative subgroups of simple algebras. In this paper we will characterize p-groups contained in full matrix algebras $M_n(\Delta)$ of fixed degree n, where Δ are division algebras of characteristic 0.

All division algebras considered in this paper will be of characteristic 0.

Let Δ be a division algebra. We will denote by $M_n(\Delta)$ the full matrix algebra of degree n over Δ. By a subgroup of $M_n(\Delta)$ we will mean a multiplicative subgroup of $M_n(\Delta)$. Further let K be a subfield of the center of Δ and let G be a finite subgroup of $M_n(\Delta)$. Now we define $V_K(G)=\{\sum\alpha_i g_i \mid \alpha_i\in K, g_i\in G\}$. Then $V_K(G)$ is clearly a K-subalgebra of $M_n(\Delta)$ and there is a natural epimorphism $KG\to V_K(G)$ where KG denotes the group algebra of G over K. Hence $V_K(G)$ is a semi-simple K-subalgebra of $M_n(\Delta)$, which is a direct summand of KG. As usual $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ denote respectively the rational number field, the real number field, the complex number field and the quaternion algebra over $\boldsymbol{R}$.

If an abelian group G has invariants $(e_1, \cdots, e_n)$, $e_n\neq 1$, $e_{i+1}\mid e_i$, we say briefly that G has invariants of length n.

We begin with

Proposition 1. *Let n be a fixed positive integer and let G be a finite abelian group. Then there is a division algebra Δ such that $G\subset M_n(\Delta)$ if and only if G has invariants of length $\leqq n$.*

Proof. This may be well known. Here we give a proof. Suppose that there is a division algebra Δ such that $G\subset M_n(\Delta)$. An abelian group G has invariants of length $\leqq n$ whenever each Sylow subgroup of G has invariants of length $\leqq n$. Hence we may assume that G is a p-group $(\neq 1)$. Let m be the length of invariants of G. Then G contains the elementary abelian group G_0 of order p^m. We can write $\boldsymbol{Q}G_0\cong\boldsymbol{Q}\oplus\overbrace{\boldsymbol{Q}(\varepsilon_p)\oplus\cdots\oplus\boldsymbol{Q}(\varepsilon_p)}^{1+p+\cdots+p^{m-1}}$ where ε_p denotes the primitive p-th root of unity. Since $V_{\boldsymbol{Q}}(G_0)$ is a direct summand of $\boldsymbol{Q}G_0$ and $G_0\subset V_{\boldsymbol{Q}}(G_0)$, we have $V_{\boldsymbol{Q}}(G_0)\cong\overbrace{\boldsymbol{Q}(\varepsilon_p)\oplus\cdots\oplus\boldsymbol{Q}(\varepsilon_p)}^{m}$. On the other hand, since