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ELLIPTIC COMPLEXES ON CERTAIN HOMOGENEOUS SPACES

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Introduction

The main purpose of this paper is to construct elliptic complexes on symmetric spaces of inner type, which are very analogous to Dolbeault complexes. When a symmetric space of the above type has a homogeneous vector bundle satisfying a certain condition, we can canonically associate to it an elliptic complex with length of half a dimension of the space, whose first term coincides with the given vector bundle (Theorem 3.1). In particular, if the symmetric space has an invariant complex structure, one can see that the elliptic complex associated in such a way is no other than the Dolbeault complex for the given holomorphic vector bundle.

In more detail, let $X=G/K$ be a symmetric space of inner type, i.e., (G, K) is a symmetric pair and rank $G=$rank K. To an irreducible K-module V, there is the homogeneous vector bundle $\mathcal{V}$ over X associated. When we denote by $\mathcal{P}$ the complex cotangent bundle over X, we define an invariant first order differential operator

$$D^0\colon C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V}\otimes\mathcal{P})$$

as the covariant differentiation induced from the invariant connection determined by the Cartan decomposition of the Lie algebra of G with respect to (G, K). Here $C^\infty(\cdot)$ denotes the space of infinitely differentiable sections of a vector bundle. It is known that this operator extends uniquely to a differential operator

$$D^q\colon C^\infty(\mathcal{V}\otimes\wedge^q\mathcal{P}) \to C^\infty(\mathcal{V}\otimes\wedge^{q+1}\mathcal{P})\,,$$

such that $D^q(s\varphi)=D^0s\wedge\varphi+sd\varphi$ for $s\in C^\infty(\mathcal{V})$, $\varphi\in C^\infty(\wedge^q\mathcal{P})$. For a lexicographical order of the root system of the complexification of the Lie algebra of G, we choose a homogeneous vector bundle $\mathcal{U}^q\subset\mathcal{V}\otimes\wedge^q\mathcal{P}$ for every $q\geqslant 1$, satisfying the following property. First it holds $D^q(C^\infty(\mathcal{U}^q))\subset C^\infty(\mathcal{U}^{q+1})$;