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LOCAL COHOMOLOGY AND CONNECTEDNESS OF ANALYTIC SUBVARIETIES

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Suppose X is an analytic subvariety in some open neighborhood G of the origin 0 in $\boldsymbol{C}^n$ with $\operatorname{codim}_{G;0}(X)=r$, where $\operatorname{codim}_{G;0}(X)$ denotes the codimension at 0 of X as a subvariety of G. Let ${}_n\mathfrak{O}$ be the structure sheaf of $\boldsymbol{C}^n$. Let $H^p_{X;0}({}_n\mathfrak{O})$, or simply $H^p_{X;0}$, denote the direct limit of $\{H^{p-1}(U-X, {}_n\mathfrak{O}) \mid U$ is an open neighborhood of 0 in $G\}$ for $p\geqq 1$. ($H^p_{X;0}$ agrees with the stalk at 0 of the sheaf defined by the p-th local cohomology groups at X with coefficients in ${}_n\mathfrak{O}$, [1], p. 79). We say that X is *locally a complete intersection* at 0 if X can be defined locally at 0 by r holomorphic functions. If X is locally a complete intersection, obviously we have

$$H^p_{X;0} = 0 \qquad \text{for} \quad p>r\,. \tag{1}$$

The question naturally arises: to what extent does (1) characterize a local complete intersection? Not much is known about the characterization of local complete intersections. In [3] Hartshorne introduces a concept of connectedness which in our case is equivalent to the following: X is *locally connected in codimension* k at 0 if the germ of X at 0 cannot be decomposed as the union of two subvariety-germs which are both different from the germ of X at 0 and whose intersection is a subvariety-germ Y with $\operatorname{codim}_{X;0}(Y)>k$. He shows that, if X is locally a complete intersection, then X is locally connected in codimension 1 at 0 (and also locally connected in codimension 1 at 0 in some properly defined formal sense). In this note we prove that (1) is a stronger necessary condition for local complete intersections than the connectedness condition. The following is our main theorem:

Theorem 1. *Suppose $q\geqq 0$. If $H^p_{X;0}=0$ for $p>q+r$, then X is locally connected in codimension $q+1$ at* 0.

For the proof of Theorem 1 we need the following:

Lemma 1. *Suppose Y is a* 1*-dimensional subvariety in some open neighborhood H of* 0 *in $\boldsymbol{C}^n$. Suppose* 0 *is the only singular point of Y and Y is locally irreducible at* 0. *Then $H^n_{Y;0}=0$.*