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ON LOCAL RIGHT PURE SEMISIMPLE RINGS OF LENGTH TWO OR THREE

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1. Introduction

We investigate in this paper non-commutative local rings R of the smallest length that are potential counter-examples to the pure semisimplicity conjecture.

Throughout the paper R is an associative ring with an identity element. We call R *local*, if the Jacobson radical $J(R)$ of R is a two-sided maximal ideal. We denote by $\mathrm{mod}(R)$ the category of finitely generated right R-modules. Given a right R-module X_R of finite length we denote by $l(X_R)$ the length of X_R.

We recall that a ring R is said to be of *finite representation type*, if R is artinian and the number of the isomorphism classes of finitely generated indecomposable right (and left) R-modules is finite. Following [24] we call a ring R *right pure semisimple*, if every right R-module is a direct sum of finitely generated modules.

It is well known that a ring R is of finite representation type if and only if R is right pure semisimple and R is left pure semisimple (see [2], [11], [18], [22]–[24]). It is still an open question, called the *pure semisimplicity conjecture*, if a right pure semisimple ring R is of finite representation type (see [2] and [24], [25], [28]). In [13] the question is answered in affirmative for rings R satisfying a polynomial identity and for self-injective rings R (see also [7], [19] and [31]). The reader is referred to [42] and to the author's expository papers [30] and [32] for a basic background and historical comments on the pure semisimplicity conjecture.

It was shown by the author in [28] and [33] that there is a chance to find a counter-example R to the pure semisimplicity conjecture and R might be hereditary with two simple non-isomorphic modules. The existence of a counter-example depends on a generalized Artin problem on division ring extensions.

In the present paper we are mainly interested in the existence of counter-examples R to pure semisimplicity conjecture that are local of the smallest length, that is, of length $l(R_R)$ two or three. This continues our study started in [28], [35] and [33].

It is shown in Lemma 3.1 that every such a local ring R has $J(R)^2 = 0$. Therefore we study representation-infinite right pure semisimple local rings R with $J(R)^2 = 0$ such that the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, R)$ is of the form $\cdots \to \bullet \to \bullet \to \bullet \to$

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