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A RELATIONSHIP BETWEEN BROWNIAN MOTIONS WITH OPPOSITE DRIFTS VIA CERTAIN ENLARGEMENTS OF THE BROWNIAN FILTRATION

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1. Introduction

Let $B = \{B_t, t \geqq 0\}$ be a one-dimensional standard Brownian motion starting from 0. To $B^{(\mu)} = \{B_t^{(\mu)} = B_t + \mu t, t \geqq 0\}$, a Brownian motion with constant drift μ, we associate the exponential additive functional

$$A_t^{(\mu)} = \int_0^t \exp\bigl(2B_s^{(\mu)}\bigr)ds, \qquad t \geqq 0,$$

which is the quadratic variation process of the corresponding geometric Brownian motion $e^{(\mu)} = \{e_t^{(\mu)} = \exp\bigl(B_t^{(\mu)}\bigr), t \geqq 0\}$.

These Wiener functionals play important roles in many fields; mathematical finance (see, e.g., Yor [33], [34], Geman-Yor [10], [11], Leblanc [17]), diffusion processes in random environment (Comtet-Monthus [5], Comtet-Monthus-Yor [6], Kawazu-Tanaka [16]), probabilistic studies related to Laplacians on hyperbolic spaces (Gruet [12], Ikeda-Matsumoto [13]) and so on. The reader will find more related topics and references in [37].

However, even for fixed t, the joint law of $\bigl(A_t^{(\mu)}, B_t^{(\mu)}\bigr)$ or, equivalently, that of $\bigl(A_t = A_t^{(0)}, B_t\bigr)$ due to the Cameron-Martin relationship between $B^{(\mu)}$ and B is fairly complicated, although it is quite tractable (see [33]). As an example of the description of this law, we present the following conditional Laplace transform ([2], [13], [18]):

$$E\left[\exp\left(-\frac{u^2 A_t}{2}\right) \,\middle|\, B_t = x\right] \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) = \int_{|x|}^{\infty} \frac{z}{\sqrt{2\pi t^3}} \exp\left(-\frac{z^2}{2t}\right) J_0(u\phi(x,z))dz,$$

where $u \geqq 0$, J_0 is the Bessel function of the first kind of order 0 and $\phi(x,z) = \sqrt{2}e^{x/2}(\cosh z - \cosh x)^{1/2}$, $z \geqq |x|$.

A quite different description of the law of $A_t^{(\mu)}$ is as follows (Geman-Yor [10], Yor [34]). Let T_λ be an exponential random variable with parameter λ, independent of B. Moreover let $Z_{1,a}$, Z_b be a Beta and a Gamma random variables with parameters