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ON HOLOMORPHIC FAMILIES OF HOLOMORPHIC MAPS

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Let D be the unit disk $\{z : |z| < 1\}$ in the complex plane $\boldsymbol{C}$ with boundary ∂D and closure $\bar{D}$, and denote by $\boldsymbol{R}$ the image of the canonical embedding $r \to r + iO$ of the real line into $\boldsymbol{C}$. The symbol ε will be used throughout to denote a complex parameter; the unit disk in the complex ε-plane will be denoted by D_p. A C^{1+a} map $\mathscr{C} : \partial D \times D_p \to \boldsymbol{D}$ $(0 < a < 1)$ is called a *holomorphic family of* C^{1+a} *curves* if

1° $\mathscr{C}_\varepsilon = \mathscr{C} | \partial D \times \{\varepsilon\}$ is a C^{1+a} Jordan curve in $\boldsymbol{C}$ for every $\varepsilon \in D_p$;

2° $\mathscr{C}_t = \mathscr{C} | \{t\} \times D_p$ is a holomorphic function for every $t \in \partial D$;

3° $\dfrac{\partial \mathscr{C}(t, \varepsilon)}{\partial t}$ is continuous in t and ε.

Denote by Ω_ε the simply-connected region in $\boldsymbol{C}$ bounded by $\mathscr{C}(\partial D \times \{\varepsilon\})$.

We are interested in the existence of holomorphic maps $f : D \times D_p \to \boldsymbol{C}$ which map $D \times \{\varepsilon\}$ conformally onto Ω_ε for every $\varepsilon \in D_p$ (f is then said to be *associated with* $\mathscr{C}$). The following theorem will be proved.

THEOREM 1. *Let $\mathscr{C} : \partial D \times D_p \to \boldsymbol{C}$ be a holomorphic family of C^{1+a} curves. If f is a holomorphic map associated with $\mathscr{C}$, then there exists a C^{1+a} homeomorphism $g : \partial D \to \partial D$ for which*

$$(*) \qquad \mathscr{C}(t, \varepsilon) = f(g(t), \varepsilon)$$

for all $(t, \varepsilon) \in \partial D \times D_p$, where f on the right hand side denotes the continuous extension of f to $\bar{D} \times D_p$.

Now $\mathscr{C}$ can always be normalized by the condition that for some $\varepsilon_0 \in D_p$, $\mathscr{C}_{\varepsilon_0}$ is the boundary value of a conformal map of $D \times \{\varepsilon_0\}$ onto Ω_{ε_0} (for let g_{ε_0} be such a conformal map, the existence of which is ensured by

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