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TOWARDS A PROBLEM IN DEFORMATIONS OF POLARIZED ALGEBRAIC $K3$ SURFACES

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(0.1) Introductory

A nonsingular algebraic surface V is called a $K3$ surface if i) $1 = p_g = \mathfrak{l}(K_V) = \dim. H^0(V, \Omega^2{}_V)$, i.e. a canonical divisor K_V on V is linearly equivalent to zero; and ii) $\mathfrak{h}^1(V) = \dim. H^1(V, \underline{O}_V) = 0$. When the characteristic is zero, condition ii) is equivalent to ii)′ $q =$ dimension of the Albanese variety of $V = 0$, and always ii) implies ii)′ as in fact $\mathfrak{h}^1(V) \geq q \geq 0$, but in non-zero characteristic it can happen that $\mathfrak{h}^1(V) > q$ ([6], [15]). When ii)′ is true, the algebraic and linear equivalences of divisors coincide on V, because of the duality between Picard and Albanese varieties of V, [8]. When i) and ii) are true, the Riemann-Roch theorem for divisors D on V reads

$$\text{(R)} \qquad \mathfrak{l}(D) - \mathfrak{h}^1(D) + \mathfrak{l}(-D) = D^{(2)}/2 + 2$$

because in general $\mathfrak{l}(D) - \mathfrak{h}^1(D) + \mathfrak{l}(K_V - D) = D^{(2)} - I(D, D + K_V)/2 + 1 + p_a(V)$ (cf. [25], ch. 4, app.) and on a $K3$ surface we have that $K_V \sim 0$, $p_g = 1$, $p_a = p_g - \mathfrak{h}^1(V) = 1$. —Here we use a standard notation: $\mathfrak{l}(D) = \dim. H^0(V, \underline{O}_V(D))$, $\mathfrak{h}^1(D) = \dim. H^1(V, \underline{O}_V(D))$, and "$\sim$" denotes linear equivalence of divisors.

Let V be a $K3$ surface. Notice that any self-intersection number $D^{(2)}$ is even, so exceptional divisors X of first kind cannot exist since, for such divisors, the fundamental cycle Z of X satisfies $Z^{(2)} = -1$ by the contractibility criterion of Castelnuovo and M. Artin, [1]; cf. [9]. On the other hand, when a surface W is ruled, $p_g(W) = 0$; so a $K3$ surface is not ruled. Then a $K3$ surface is always a minimal model of

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