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# CLASS-NUMBER PROBLEMS FOR CUBIC NUMBER FIELDS

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## 1. Introduction

Let $\mathbf{M}$ be any number field. We let $D_{\mathrm{M}}$, $d_{\mathrm{M}}$, $h_{\mathrm{M}}$, $\zeta_{\mathrm{M}}$, $\mathbf{A}_{\mathrm{M}}$ and $\mathrm{Reg}_{\mathrm{M}}$ be the discriminant, the absolute value of the discriminant, the class-number, the Dedekind zeta-function, the ring of algebraic integers and the regulator of $\mathbf{M}$, respectively. We set $c = \dfrac{3+2\sqrt{2}}{2}$. If $q$ is any odd prime we let $(\cdot/q)$ denote the Legendre's symbol. We let $D_P$ and $d_P$ be the discriminant and the absolute value of the discriminant of a polynomial $P$.

LEMMA A (See [Sta, Lemma 3] and [Hof, Lemma 2]). *Let $\mathbf{M}$ be any number field. Then, $\zeta_{\mathrm{M}}$ has at most one real zero in*

$$\left[1 - \frac{1}{c \log d_{\mathrm{M}}}, 1\right[;$$

*if such a zero exists, it is simple and is called a Siegel zero.*

LEMMA B (See [Lou 2]). *Let $\mathbf{M}$ be a number field of degree $n = r_1 + 2r_2$ where $\mathbf{M}$ has $r_1$ real conjugate fields and $2r_2$ complex conjugate fields. Let $s_0 \in [(1/2), 1[$ be such that $\zeta_{\mathrm{M}}(s_0) \leq 0$. Then,*

$$\mathrm{Res}_{s=1}(\zeta_{\mathrm{M}}) \geq (1-s_0) d_{\mathrm{M}}^{(s_0-1)/2}\left(1 - \frac{2r_1}{d_{\mathrm{M}}^{s_0/2n}} - \frac{2\pi r_2}{d_{\mathrm{M}}^{s_0/n}}\right).$$

## 2. Lower bounds for class-numbers of cubic number fields

Let $\mathbf{K}$ be a cubic number field. If $\mathbf{K}/\mathbf{Q}$ is normal then $\mathbf{K}$ is a cyclic cubic number field. Let $f_{\mathbf{K}}$ be its conductor. Then $d_{\mathbf{K}} = f_{\mathbf{K}}^2$ and $\zeta_{\mathbf{K}}(s) = \zeta(s)L(s, \chi)$ $L(s, \bar{\chi})$ where $\chi$ is a primitive cubic Dirichlet character modulo $f_{\mathbf{K}}$. Hence, we get

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