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Nagoya Math. J.
Vol. 144 (1996), 137–154

EXPONENTIAL ASYMPTOTICS IN THE SMALL PARAMETER EXIT PROBLEM

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1. Introduction

Let $\mathcal{M}$ be a d-dimensional Riemannian manifold of class C^∞ with Riemannian metric $g = (g_{ij})$ and let D be a connected domain in $\mathcal{M}$ having a non-empty smooth boundary ∂D and a compact closure $\bar{D}$. Suppose that $b^\varepsilon \in \mathfrak{X}(\mathcal{M}) = \{C^\infty\text{-vector fields on } \mathcal{M}\}$, $\varepsilon > 0$, are given and that $\{b^\varepsilon\}$ converges uniformly to $b \in \mathfrak{X}(\mathcal{M})$ on D' as $\varepsilon \downarrow 0$ for some neighborhood D' of D. Consider the diffusion process (x_t^ε, P_x) on D' with a small parameter $\varepsilon > 0$ generated by

$$\mathcal{L}^\varepsilon = \frac{\varepsilon^2}{2}\Delta + b^\varepsilon, \tag{1.1}$$

where Δ is the Laplace-Beltrami operator on $\mathcal{M}$. Uniqueness of the process requires some boundary condition on $\partial D'$. However boundary conditions are not mentioned since the process is considered only before the time when it leaves a small neighborhood of $\bar{D}$. In this paper, we shall study the asymptotic behavior of the expectation of the first exit time τ^ε from the domain D; i.e.,

$$\tau^\varepsilon = \inf\{t > 0\,; x_t^\varepsilon \notin D\},$$

under the following assumptions:

(A_1) (gradient condition) there exists a potential function $U \in C^\infty(\bar{D})$ such that $b = -\dfrac{1}{2}\operatorname{grad} U$ on $\bar{D}$;

(A_2) the set of critical points $\mathscr{C} = \{x \in D\,; \operatorname{grad} U(x) = 0\}$ consists of finite number of connected components $K_1, \ldots, K_l$ (each of which is called compactum) such that, for arbitrary two points $x, y \in K_i$, there is an

Received July 12, 1993.
Revised June 23, 1995.