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The C-numerical range of a 3×3 normal matrix

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Abstract In this note we study the shape of the C-numerical range of a 3×3 normal matrix.

1. Introduction and Results

In this decade many authors obtained new results in numerical ranges, numerical radii of linear operators and their related topics (cf. [2, 3, 6, 7, 9, 11]). In the paper [5] the author studied a special case of the C-numerical ranges. Recent work [4] provides us a new method to treat the C-numerical ranges. We will prove "weak convexity" of the C-numerical ranges in some sense.

Suppose that $C=\mathrm{diag}(c_1,c_2,c_3)$ and $T=\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3)$ are complex 3×3 diagonal matrices. We consider a compact subset $W_C(T)$ of the Gaussian plane $\mathbf{C}$ defined by

$$W_C(T)=\{\mathrm{tr}(C\,UTU^*): U\in M_3(\mathbf{C}),\ U^*U=UU^*=I_3\}. \tag{1.1}$$

However this range is not necessarily convex, this range is star-shaped with respect to the point

$$(1/3)(c_1+c_2+c_3)(\alpha_1+\alpha_2+\alpha_3)\in W_C(T). \tag{1.2}$$

We consider the following 6 special points of $W_C(T)$:

$$\sigma_1 = c_1\alpha_1+c_2\alpha_2+c_3\alpha_3,\ \sigma_2 = c_1\alpha_2+c_2\alpha_3+c_3\alpha_1,\ \sigma_3 = c_1\alpha_3+c_2\alpha_1+c_3\alpha_2, \tag{1.3}$$

$$\sigma_4 = c_1\alpha_1+c_2\alpha_3+c_3\alpha_2,\ \sigma_5 = c_1\alpha_3+c_2\alpha_2+c_3\alpha_1,\ \sigma_6 = c_1\alpha_2+c_2\alpha_1+c_3\alpha_3. \tag{1.4}$$

These are called σ-points of the range $W_C(T)$. The 9 line segments