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CONVEXITY PRESERVING FOR FULLY NONLINEAR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS*

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Dedicated to Professor Neil Trudinger on the occasion of his 65th birthday

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1. Introduction. In this paper, we consider the question of the preservation of convexity of the Cauchy problem for fully nonlinear integro-differential equation

$$u_t = F(\nabla^2 u, \nabla u, u, x, t) + \mathbb{B}u, \quad (x,t) \in \mathbb{R}^n \times [0,T], \tag{1.1}$$

where $F = F(r,p,u,x,t)$ is a given function in $\Gamma = \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times [0,T]$, $\mathcal{S}^n$ denotes the space of real symmetric $n \times n$ matrices, and $\mathbb{B}u$ is a integro-differential operator as follows

$$\mathbb{B}u = \lambda(t) \int_0^1 (u(x+\psi(x,t,\eta),t) - u(x,t) - \psi(x,t,\eta)\cdot\nabla u(x,t))d\eta, \tag{1.2}$$

here $\lambda(t)$ is a given nonnegative function, $\psi(x,t,\eta)$ is a given function in $\mathbb{R}^n \times [0,T] \times [0,1]$.

Equations (1.1) are second order fully nonlinear integro-differential equations of parabolic type. These equations are derived from the pricing problem of financial derivatives and optimal portfolio selection problem in a market where underlying assets prices are modeled by a Lévy process $S(\tau)$ (Chapter 9 in [14], see also [6], [16] and [3]). A typical example for the European option pricing problem in one-dimensional is as follows. Let $(W_\tau)_{\tau\geq 0}$ be the standard Brownian motion, $(N_\tau)_{\tau\geq 0}$ be Poisson process with parameter λ and $(U_j)_{j\geq 1}$ be a sequence of square integrable independent, identically distributed random variables, with values in $(-1,+\infty)$. Assume the Lévy process $S(\tau)$ evolves according to the following stochastic differential equation

$$\mathrm{d}S(\tau) = S(\tau)(\mu \mathrm{d}\tau + \sigma \mathrm{d}W_\tau + \mathrm{d}(\sum_{j=1}^{N_\tau} U_j)), \tag{1.3}$$

where μ, σ are the drift and volatility respectively. Furthermore, we assume the processes $(W_\tau)_{\tau\geq 0}$, $(N_\tau)_{\tau\geq 0}$, $(U_j)_{j\geq 1}$ are independent. Let $\hat{p}(\xi)$ be the probability density function of random variable U_1, thus price function $V(s,\tau)$ of European option with finite horizon T satisfies the following linear equation

$$\frac{\partial V}{\partial \tau} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-\lambda k)s\frac{\partial V}{\partial s} - (r+\lambda)V + \lambda\int_{-1}^{\infty} V(s(1+\xi),\tau)\hat{p}(\xi)d\xi = 0, \tag{1.4}$$

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