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LOWER BOUNDS FOR THE DEFICIENCY INDICES OF $T_0(\tau^+)T_0(\tau)$, WHERE τ IS A LINEAR ORDINARY DIFFERENTIAL EXPRESSION

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Abstract. Let τ be a linear ordinary differential expression with smooth coefficients, defined on an interval $I \subseteq \mathbb{R}$, and let τ^+ denote its formal adjoint. We consider the nonnegative, symmetric operator $T_0(\tau^+)T_0(\tau)$ operating in $L^2(I)$ and show that the dimensions of its deficiency spaces (which are equal) are bounded below by $2(\dim\ker T_1(\tau^+) - \dim\ker T_0(\tau) + \dim\ker T_1(\tau) - \dim\ker T_0(\tau^+))$, and that the dimensions of the deficiency spaces equal this lower bound when $0 \notin \sigma_e(\tau)$, the essential spectrum of τ, equivalently, when $0 \notin \sigma(\tau^+\tau)$. From this result, we develop other lower bounds for the dimensions of the deficiency spaces of $T_0(\tau^+)T_0(\tau)$.

Here is some notation, briefly. I denotes an interval of the real line $\mathbb{R}$. Let τ be a linear ordinary differential expression with $C^\infty(I)$ coefficients, of integer order $n > 0$, which is defined on I, and let τ^+ denote its formal adjoint. For any linear ordinary differential expression ζ, let $T_0(\zeta)$ denote its closed minimal operator and let $T_1(\zeta)$ denote its (closed) maximal operator. We denote by $\sigma_e(\zeta)$ the *essential spectrum* of $T_1(\zeta)$. If it is important to specify the interval, we write $T_i(\zeta, I)$, $i = 0, 1$, or $\sigma_e(\zeta, I)$. For any operator T, $\mathcal{D}(T)$ denotes its domain. If S is a densely defined symmetric operator whose adjoint is S^*, let $\mathcal{D}_\pm(S^*) = \{f \in \mathcal{D}(S^*) : S^*(f) = \pm if\}$.

Our goal is information about $\dim \mathcal{D}_\pm(T_1(\tau\tau^+))$, granted information about $T_1(\tau^+\tau)$ and related operators. Our method is to estimate

$$d_\pm = \dim\{f \in \mathcal{D}_\pm(T_1(\tau^+\tau)) : \tau(f) \in L^2(I)\}. \tag{1}$$

This is pertinent since $\tau^+\tau(f) = \pm if$ and $\tau(f) \in L^2(I)$ entail $\tau(f) \in \mathcal{D}_\pm(T_1(\tau\tau^+))$ (that $f, \tau(f) \in C^\infty(I)$ follows from the coefficients of τ being in $C^\infty(I)$ [1, XIII.1.4]). We prove:

First, that $0 \notin \sigma_e(\tau^+\tau)$ entails $0 \notin \sigma_e(\tau^+)$ (whence $0 \notin \sigma(\tau^+\tau)$, if and only if $0 \notin \sigma_e(\tau^+)$ if and only if $0 \notin \sigma(\tau)$).

Second, we extend to arbitrary intervals a formula of Kauffman, Read, and Zettl [7, II.4.4]:

$$\begin{aligned}\dim \mathcal{D}(T_1(\tau))/\mathcal{D}(T_0(\tau)) \geq{}& \dim\ker T_1(\tau^+) - \dim\ker T_0(\tau)\\ &+ \dim\ker T_1(\tau) - \dim\ker T_0(\tau^+),\end{aligned}$$

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