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On the Solvability of Painlevé II and IV

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Abstract. We introduce a rigorous methodology for studying the Riemann-Hilbert problems associated with certain integrable nonlinear ordinary differential equations. For concreteness we investigate the Painlevé II and Painlevé IV equations. We show that the Cauchy problems for these equations admit in general global, meromorphic in t solutions. Furthermore, for special relations among the monodromy data and for t on Stokes lines, these solutions are bounded for finite t.

1. Introduction

Flaschka and Newell [1] and Jimbo, Miwa, and Ueno [2] have introduced a powerful approach for studying the initial value problem of certain nonlinear ODE's: They have shown that solving such an initial value problem is essentially equivalent to solving an inverse problem for an associated isomonodromic linear equation. This inverse problem can be formulated in terms of monodromy data which can be calculated from initial data. Fokas and Ablowitz [3] have shown that the inverse problem can be formulated as a matrix, singular, discontinuous Riemann-Hilbert (RH) problem defined on a complicated contour. Hence techniques from RH theory can be employed to study the solvability of certain nonlinear ODE's. The above method, which is an extension of the inverse scattering transform method, is called inverse monodromic transform (IMT), and can be thought of as a nonlinear analogue of the Laplace's method for solving linear ODE's.

The six Painlevé transcendents, P I–P IV, are the most well known nonlinear ODE's that can be studied using the IMT method. P II has been studied in [1, 3], a special case of P III in [1], P IV and P V in [4]. We refer the interested reader to [4] for a historical perspective on the Painlevé equations. Here we only note that these equations: (a) have the Painlevé property, i.e. their solutions are free from movable critical points [5], (b) possess particular solutions which are either rational or can