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Homothetic and Conformal Symmetries of Solutions to Einstein's Equations

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Abstract. We present several results about the nonexistence of solutions of Einstein's equations with homothetic or conformal symmetry. We show that the only spatially compact, globally hyperbolic spacetimes admitting a hypersurface of constant mean extrinsic curvature, and also admitting an infinitesimal proper homothetic symmetry, are everywhere locally flat; this assumes that the matter fields either obey certain energy conditions, or are the Yang–Mills or massless Klein–Gordon fields. We find that the only vacuum solutions admitting an infinitesimal proper conformal symmetry are everywhere locally flat spacetimes and certain plane wave solutions. We show that if the dominant energy condition is assumed, then Minkowski spacetime is the only asymptotically flat solution which has an infinitesimal conformal symmetry that is asymptotic to a dilation. In other words, with the exceptions cited, homothetic or conformal Killing fields are in fact Killing in spatially compact or asymptotically flat spactimes. In the conformal procedure for solving the initial value problem, we show that data with infinitesimal conformal symmetry evolves to a spacetime with full isometry.

1. Introduction

Virtually all explicitly known spacetime solutions of Einstein's equations admit some nontrivial isometry group. This is not surprising since the Einstein equations are very difficult to solve, and isometries simplify them considerably. While much physical insight on astrophysical and cosmological questions has been obtained from the study of spacetimes with lots of symmetry, it clearly would be useful to examine solutions with a smaller isometry group, or even a trivial one. One possible way for reducing spacetime symmetry without giving up all the simplifications it provides is to replace spacetime isometries with spacetime *conformal symmetries or homothetic symmetries.* A conformal symmetry preserves the metric up to a general point-dependent scale factor, while for a homothetic symmetry the scale factor must be constant.