Commun. Math. Phys. 105, 461–472 (1986)

Communications in Mathematical Physics

A New Integrable Case of the Motion of the 4-Dimensional Rigid Body

A. G. Reyman and M. A. Semenov-Tian-Shansky

Leningrad Branch of V. A. Steklov Mathematical Institute, SU-191011 Leningrad, USSR

Abstract. A Lax pair for a new family of integrable systems on SO(4) is presented. The construction makes use of a twisted loop algebra of the G_2 Lie algebra. We also describe a general scheme producing integrable cases of the generalized rigid body motion in an external field which have a Lax representation with spectral parameter. Several other examples of multi-dimensional tops are discussed.

Introduction

Starting with the work of Arnold [1], the study of multi-dimensional tops has already a rather long history with the well-known paper of Manakov [2] as one of its highlights. In recent years much effort was spent on the classification of integrable cases. The case of the four-dimensional top is particularly interesting since it has direct physical significance [3]. Some of the integrable cases discovered recently are new even in the classical setting, e.g. the motion of a top in an arbitrary quadratic potential [4] (This case was considered earlier, along with several others, in [5] but passed practically unnoticed. Another interesting system indicated in [5], a pair of interacting tops, was rediscovered in [6]).

In this paper we shall describe a new integrable case of the four-dimensional top. Our construction is based on the so-called Kostant-Adler scheme and on the use of affine Lie algebras. This technique has already been applied to the study of multi-dimensional tops and related systems in [7, 5] and independently in [8, 9]. Our main technical tool consists in twisting the loop algebra of a simple Lie algebra by a Cartan automorphism which leads precisely to Hamiltonian systems of the generalized rigid body type. (This was already indicated in [7] but was missed in [8, 9].)

Our principal example is connected with the split real form of the G_2 simple Lie algebra. Recently Adler and van Moerbeke [10] announced a classification of left-invariant metrics on SO(4) that are algebraically completely integrable. Our example fits into their list, thus providing a Lax pair for the last case of the