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The Geometry of Real Sine-Gordon Wavetrains

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Abstract. The characterization of *real*, N phase, quasiperiodic solutions of the sine-Gordon equation has been an open problem. In this paper we achieve this result, employing techniques of classical algebraic geometry which have not previously been exploited in the soliton literature. A significant by-product of this approach is a natural *algebraic* representation of the full complex isospectral manifolds, and an understanding of how the real isospectral manifolds are embedded. By placing the problem in this general context, these methods apply directly to all soliton equations whose multiphase solutions are related to hyperelliptic functions.

Introduction

The N phase quasiperiodic solutions of the Korteweg-de Vries (KdV), sine-Gordon (sG), and sinh-Gordon (sh-G) equations have explicit representations in terms of theta functions of N variables [7, 8]. The exact integration for this class of solutions is achieved through the spectral theory of an associated linear system: the nonlinear evolution equation generates one of an infinite family of involutive isospectral flows for the x-eigenvalue problem. An important distinction, however, is that the spectral problem for *real N phase potentials* is selfadjoint for KdV and sh-G, but non-selfadjoint for sine-Gordon. The methods in [14] for KdV and in [5] for sh-G are not amenable to non-selfadjoint problems, so the real N phase sine-Gordon theory remained open.

The point of view taken here is that the formal inverse spectral structure of all these partial differential equations (p.d.e.'s) is the same at the level of complex quasiperiodic potentials. We deduce the structure of the real isospectral class as a subset of the full complex isospectral class. This common structure of N phase potentials consists of: N dynamical variables, $\boldsymbol{\mu}=(\mu_1, \ldots, \mu_N)$, which are Dirichlet

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