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The Dirac Equation in Exterior Form

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Abstract. Using the correspondence between the Clifford and exterior algebras we write the Dirac equation in terms of differential forms. The covariances of the theory are then examined. We show in detail the correspondence with usual matrix methods.

Introduction

Kähler [1] has used a correspondence between Clifford and exterior algebras associated with space-time to describe particles with half integer spin by means of sections of the exterior bundle over space time (inhomogeneous differential forms). In Minkowski space Kähler's equation decouples into four minimal left ideals of the Clifford algebra, and is equivalent to four identical Dirac equations [2, 3]. Thus the Kähler equation is not the Dirac equation. Moreover, in an arbitrary space-time the Kähler equation does not split into minimal left ideals. These features of the Kähler equation are not inevitable consequences of using differential forms for the description of half-integer spin, indeed we here show how the Kähler equation can be modified so that, even in a curved space-time, it describes fields lying in one minimal left ideal. In this way we are lead to the Dirac equation.

We shall make contact with the usual matrix formulation of the Dirac equation by making clear how, and in what sense, the Clifford algebra is a matrix algebra. Since we regard the Clifford algebra as being embedded in the Kähler-Atiyah algebra, a basis for this matrix algebra will consist of differential forms. This differentiable basis is the main feature inherited from exploitation of the Kähler-Atiyah algebra. It will be shown how covariant derivatives of this basis can provide the usual "spin connection" terms in the curved space Dirac equation.

Having written the Dirac equation in terms of inhomogeneous differential forms it is natural to question the compatability of the "spinorial" nature of the equation with the "tensorial" nature of the forms. We shall argue that there are two a priori distinct covariances; a trivial GL(4, R) frame covariance, and a GL(4, R)