Commun. math. Phys. 2, 133—146 (1966)

Dirac Matrices and the Dirac Matrix Description of Lorentz Transformations*

A. J. MACFARLANE

Department of Physics, Syracuse University, Syracuse, New York

Received December 5, 1965

Abstract. This paper gives what is believed to be a new discussion of Dirac matrices and of the Dirac matrix description of Lorentz transformations. The five anticommuting quantities γ^J ($J = 1, 2, 3, 0, 5$) are treated on an equal footing and recognition of the rule for expressing the three-fold product $\gamma^J \gamma^K \gamma^L$ in terms of one- and two-fold products and of invariant "five-space" tensors g^{JK}, ε^{JKLPQ} allows all kinds of multiplication and trace laws for Dirac matrices to be derived systematically. The "five-space" formalism for Dirac matrices affords a very convenient vehicle for the Dirac matrix description of de Sitter transformations of a space with quadratic form $g_{KL}x^K x^L$. By considering the subset of these which leave the coordinate x^5 invariant, the Dirac matrix description of Lorentz transformations is obtained. Not only does this description give the well-known formula for any Lorentz transformation matrix L in terms of the matrix S, which enters the transformation law of a Dirac spinor $\psi(x)$ under L, it also gives an explicit and apparently new inverse formula expressing S in terms of L.

1. Introduction

In this paper, we present what is believed to be a new discussion of Dirac matrices and of the Dirac matrix description of Lorentz transformations.

Our discussion of Dirac matrices differs from the standard one[1] in that it treats on an equal footing the five anticommuting quantities $\gamma^J \equiv (\gamma^\mu, \gamma^5)$ and gives their defining relations in the form [5]

$$\gamma^J \gamma^K + \gamma^K \gamma^J = 2g^{JK} \tag{1.1}$$

$$\gamma^{J+} = \gamma^0 \gamma^J \gamma^0 \tag{1.2}$$

$$\gamma^J \gamma^K \gamma^L = g^{JK}\gamma^L - g^{JL}\gamma^K + g^{KL}\gamma^J - \frac{1}{2}\varepsilon^{JKLPQ}\gamma_P \gamma_Q\,, \tag{1.3}$$

* Research supported in part by the U.S. Atomic Energy Commision under Contract AT (30-1) 3399.

[1] Most of the algebraic properties of Dirac matrices were obtained first by PAULI [1] and are reviewed in [2]. Alternatively one may refer to any of a large number of standard works on relativistic quantum mechanics and field theory, e.g. [3] and [4].