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COUNTABLE TIGHTNESS AND PROPER FORCING

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One of the most basic and natural generalizations of first countability is countable tightness: the condition that, whenever x is in the closure of A, there is a countable subset B of A such that $x \in \overline{B}$. Countably tight spaces include sequential spaces, i.e., those in which closure is obtainable by iteration of the operation of taking limits of convergent sequences. The two classes are distinct, since there are easy examples of countable, nondiscrete spaces with only trivial convergent sequences. On the other hand, it was long a major unsolved problem whether every compact Hausdorff space of countable tightness is sequential. First posed in [**1**] and motivated by the main results of [**2**], it gained importance from subsequent discoveries on the strong structural properties enjoyed by compact sequential spaces (see [**3**] and its references). A negative answer was shown to be consistent by Ostaszewski [**4**], who used Gödel's Axiom of Constructibility ($V = L$) to construct a countably compact space X whose one-point compactification is countably tight; of course, no sequence from X can converge to the extra point. Now (Theorem 2) we have shown that a positive answer follows from the Proper Forcing Axiom (PFA), introduced in [**5**]. Our research has uncovered many other striking consequences of PFA, numbered below. None was known to be consistent until now, nor was the following consequence of Corollary 1 and Theorem 3: the ordinal space ω_1 embeds in every first countable, countably compact, noncompact T_2 space (in particular, in every countably compact nonmetrizable T_2 manifold), assuming PFA. This remains true if "first countable" is weakened to "character $\leq \omega_1$," meaning every point has a local base of cardinality $\leq \omega_1$. This leads to a remarkable structure theorem for regular, countably compact spaces of character $\leq \omega_1$ (e.g. the product space $[0,1]^{\omega_1}$): under PFA, the closure of every set A can be taken by first adjoining all limits of convergent sequences, and then adjoining to the resulting set $A^\frown$ all points x for which there is a copy W of ω_1 in $A^\frown$ such that $W \cup \{x\}$ is homeomorphic to $\omega_1 + 1$.

Although large cardinals are needed to prove the consistency of PFA, all our PFA results are consistent if ZF is consistent. This is established by using ω_2-p.i.c. [**6**, Chapter VIII] posets and a ground model with a $\diamond_{\omega_2}$-sequence to capture approximations to possible counterexamples in a countable support

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