BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

ONE-SIDED INEQUALITIES FOR THE SUCCESSIVE DERIVATIVES OF A FUNCTION

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Communicated by R. K. Miller, November 21, 1975

We fix an integer $n \geqslant 2$ and consider the class F of bounded continuous functions on the positive x-axis satisfying

(i) $-1 \leqslant f(x) \leqslant 1$ for $x \in R^+$,

(ii) $f^{(n-1)}(x)$ is absolutely continuous on R^+,

(iii) $f^{(n)}(x) \leqslant 1$ a.e. on R^+.

Under these conditions on f, our goal is to establish best possible inequalities for the intermediate derivatives $f^{(j)}(x)$. We thus extend the work begun by Landau, further developed by Schoenberg and Cavaretta, and by Hörmander [4], [5], [2].

To settle our question for the class F, we need, as extremal functions, the monosplines of R. S. Johnson. A monospline of degree n with k knots is a function of the form

$$M(x) = \frac{x^n}{n!} + \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{k} c_i (x - \xi_i)_+^{n-1}$$

where the a_i, c_i, and ξ_i are freely chosen real parameters. We note that $M^{(n-1)}(x)$ consists of $k + 1$ straight line segments, each of slope 1. Considering such monosplines restricted to $[-1, 1]$, Johnson [3] proves the following

THEOREM. *There exists a uniquely determined monospline $M_{n,k}(x)$ having precisely $n + 2k + 1$ points of equioscillation on $[-1, 1]$. In addition, $M_{n,k}$ has least* sup norm *on $[-1, 1]$; i.e., $\|M_{n,k}\|_\infty \leqslant \|M\|_\infty$ with equality only if $M = M_{n,k}$.*

The relevance of these functions to our class F becomes apparent after we make a preliminary change of scale and origin. We consider $f(x) = aM_{n,k}(bx)$ $(a > 0, b > 0)$ and determine $a = a_{n,k}$ and $b = b_{n,k}$ so that $\|f\|_\infty = 1$ on $[-b^{-1}, b^{-1}]$ and $f^{(n)}(x) = 1$ except at its knots. Then define $B_{n,k}(x) = aM_{n,k}(bx - 1)$ on the interval $[0, 2b^{-1}]$. In this fashion we obtain the monospline $B_{n,k}$ which on $[0, 2b^{-1}]$ is of norm 1 and has precisely $n + 2k + 1$ points of equioscillation there. By elementary zero counting arguments, one readily verifies that $\operatorname{sign} B_{n,k}^{(j)}(0) = (-1)^{n+j}$.

With these preliminaries, we can now state the main theorem and its corollary.

AMS (MOS) subject classifications (1970). Primary 41A15; Secondary 46B10.