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THE MORSE-PALAIS LEMMA ON BANACH SPACES

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1. **Introduction.** In [3], Tromba gave a definition of nondegeneracy and generalized the well-known Morse-Palais lemma (see [1], [2]) to a real Banach space which is a dual space. In [4], he gave a stronger definition of nondegeneracy and proved the corresponding theorem for "any" Banach space. However, his definition of nondegeneracy in [4] implies that if the Banach space under consideration is separable then its dual space is also separable (see line 4, p. 86). Consequently, his results are not applicable to the Banach space of continuous functions on [0, 1] with sup norm. In this note we define nondegeneracy (and strong nondegeneracy) in a simpler way and prove the Morse-Palais lemma actually for any Banach space. An example is also given. We point out that our conditions for nondegeneracy are so weak that our theorem is a generalization of Palais' theorem even if the Banach space is a Hilbert space. (We do not require the invertibility of $D^2f(p)$ at a critical point p.) We remark that Uhlenbeck's definition of weak nondegeneracy [5] has no apparant relation with ours.

2. **Definitions and theorem.** Let f be a C^k function ($k\geqq 1$) defined on an open set U in a (real) Banach space B. $p\in U$ is a critical point of f if $Df(p)=0$. We will always regard $D^2f(x)$ and $D^3f(x)$ as elements of $L(B, B^*)$ and $L(B, L(B, B^*))$, respectively.

DEFINITION. Let f be at least C^2. The critical point p is *nondegenerate* if $D^2f(p)$ is injective and there exists a neighborhood $W\subset U$ of p such that (1) $D^2f(x)(B)\subset D^2f(p)(B)$ for all $x\in W$, (2) $D^2f(p)^{-1}\circ D^2f(x)\in L(B, B)$, and (3) the map $x\mapsto D^2f(p)^{-1}\circ D^2f(x)$ is continuous from W into $L(B, B)$ (operator norm topology for $L(B, B)$).

REMARK. In general, $D^2f(p)^{-1}$ is not a bounded operator in any reasonable sense. What we require in (2) is that the well-defined map $D^2f(p)^{-1}\circ D^2f(x)$ is a bounded operator of B. Note also that when B is a Hilbert space the invertibility of $D^2f(p)$ implies nondegeneracy (also the following strong nondegeneracy if f is C^3).

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