BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 1, January 1972

FUNCTIONAL COMPOSITION ON SOBOLEV SPACES

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Communicated by James Serrin, July 23, 1971

Introduction. This announcement concerns the study of conditions on functions $g(x, t_1, \ldots, t_m)$, $x \in \Omega \subset R_n$, under which the function g provides, via composition, a mapping from certain product Sobolev spaces $W_{r,q}(\Omega)^m$ to other Sobolev spaces $W_{1,p}(\Omega)$. Such questions are of interest in the study of nonlinear partial differential equations and elsewhere (see **[4]**, for example). Much of the analysis hinges on the use of a (seemingly new) notion of absolute continuity on tracks of absolutely continuous curves. This notion may also be useful elsewhere.

Statement of results. In what follows $\mathscr{H}_1$ denotes one-dimensional Hausdorff measure, $\mathscr{L}_n$ denotes n-dimensional Lebesgue measure, and R_k denotes k-dimensional Euclidean space.

Let $T \subset R_m$ be the track of an absolutely continuous curve. Then by results of Roger **[5]** and Federer **[1]**, for all points $y \in T$ with the exception of an $\mathscr{H}_1$-null set there is a unique pair of tangent directions to T at y, to be denoted by unit vectors θ_y, $-\theta_y$. Let $g: T \to R_1$ be defined on T. We say that g has a tangential derivative at y if, at y, T possesses a unique pair of tangent directions and if both the following relations hold for sequences $\{y_i\} \in T$:

$$\frac{\overline{yy_i}}{|y_i - y|} \to \theta_y \Rightarrow \frac{g(y_i) - g(y)}{|y_i - y|} \to \Lambda,$$

$$\frac{\overline{yy_i}}{|y_i - y|} \to -\theta_y \Rightarrow \frac{g(y_i) - g(y)}{|y_i - y|} \to -\Lambda, \qquad \Lambda \in R_1.$$

In this case the quantity $|\Lambda| \equiv D_T g(y)$ is called *the tangential derivative* of g at y. We shall say that g is *absolutely continuous* on T provided that $D_T g$ is defined $\mathscr{H}_1$-a.e. on T (in this case $D_T g$ is necessarily $\mathscr{H}_1$-measurable) and the following relation is satisfied

$$\text{(ac)} \qquad |g(y) - g(y')| \leqq \int_U D_T g(y)\, d\mathscr{H}_1 < \infty,$$

whenever $y, y' \in T$ and $U \subset T$ is a closed connected subset containing y and y'. Call a function $g: R_m \to R_1$ *fully absolutely continuous* provided

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AMS 1969 *subject classifications.* Primary 3538, 2880; Secondary 4638.
Key words and phrases. Sobolev spaces, absolute continuity on tracks, full absolute continuity, chain rule, total differential, cone property, locally Lipschitz.
[1] Partially supported by NSF Grants GP 24339 and GP 28377.