ON REPRESENTATIONS ASSOCIATED WITH SYMMETRIC SPACES

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1. **Introduction.** The results of [2] on the structure of the symmetric algebra or universal enveloping algebra over a complex reductive Lie algebra, as a module for the adjoint group, are generalized to the symmetric space case. In particular one obtains a separation of variables theorem (freeness over the ring of invariants). Also multiplicities of the various representations are given as well as the degrees of the homogeneous subspaces on which they occur. We use the notation of [3].

2. **Sections and the principal normal TDS.**

2.1. Now if $\mathfrak{u}$ is a principal normal TDS in $\mathfrak{g}$ then one can show that any nonzero element in $\mathfrak{u}\cap\mathfrak{s}$ is necessarily regular. In fact up to a scalar it is K-conjugate to the unique element $w\in\mathfrak{g}$ defined by the relations

(1) $w\in\mathfrak{a}\cap[\mathfrak{g},\mathfrak{g}]$ and

(2) $\langle w, \alpha_i\rangle=1$ where $\{\alpha_1, \cdots, \alpha_d\}=\Sigma$ is the set of simple roots. In particular w can be embedded in a principal normal TDS $\mathfrak{u}$.

Let such a $\mathfrak{u}$ be fixed. Then $\mathfrak{u}$ has as a basis a principal normal S-triple (x, e, f) where $w=(e+f)/2$.

Now let $\tilde{\mathfrak{g}}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{a}$ and $\mathfrak{u}$.

THEOREM 1. *$\tilde{\mathfrak{g}}$ is a reductive Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{a}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$. (Also $\tilde{\mathfrak{g}}$ is semisimple in case $\mathfrak{g}$ is semisimple). Moreover the roots $\Delta\subseteq\mathfrak{a}'$ of $\tilde{\mathfrak{g}}$ is exactly the subset $\Lambda^1\subseteq\Lambda$ of all restricted roots $\phi\in\Lambda$ such that $\phi/2$ is not a root. Furthermore the Weyl group of* $(\tilde{\mathfrak{g}}, \mathfrak{a})$ *is just the Weyl group of $\mathfrak{g}$ associated with $\mathfrak{a}$* ("baby Weyl group").

Finally $\mathfrak{u}$ is a principal TDS of $\tilde{\mathfrak{g}}$ in the sense of [1].

REMARK 1. If $\mathfrak{u}$ is chosen so that $\mathfrak{u}$ is the complexification of $\mathfrak{u}\cap\mathfrak{g}_R$ (and it can be so chosen) then $\tilde{\mathfrak{g}}$ is the complexification of $\tilde{\mathfrak{g}}_R=\tilde{\mathfrak{g}}\cap\mathfrak{g}_R$ and $\tilde{\mathfrak{g}}_R$ is the normal real form of $\tilde{\mathfrak{g}}$. That is $\tilde{\mathfrak{g}}$ is defined and split over $\boldsymbol{R}$. (It is a maximal such subalgebra of $\mathfrak{g}$.)

2.2 Now in [2] a cross-section was found for the set of all regular elements (called quasi-regular in [2] but henceforth called regular since the case at hand here generalizes the case in [2]). In fact by Theorem 8 in [2] $f+\tilde{\mathfrak{g}}^e$ is a cross-section of the set of all regular elements in $\tilde{\mathfrak{g}}$. On the other hand if we consider the complex Cartan