ISOMETRIES OF ORLICZ SPACES

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The purpose of the present note is to sketch a solution for the problem of determining the form of all isometries of any reflexive Orlicz space.[1] A partial result in that direction was obtained earlier by J. Lamperti [4] (who suggested this problem to us recently). The ideas of the proof are very closely related to those used recently by the author to develop a unified and slightly extended theory (unpublished) [6] for the classical results of Banach [1], Stone [8] and Kadison [2] (see also [4]) on isometries of $C(X)$, L_p spaces, and C^* algebras. The systematic use of semi-inner-product spaces, and generalized hermitians [5], plays a central role. A semi-inner-product space, is a vector space X on which there is defined a (complex valued) form $[x, y]$ satisfying:

(i) Linearity in x,

(ii) $[x, x] > 0$ if $x \neq 0$,

(iii) $|[x, y]|^2 \leqq [x, x][y, y]$.

X is then normed under $\|x\| = [x, x]^{1/2}$.

From now on, X is a reflexive Orlicz space [7; 3] whose unit sphere is the set $\{f \in X: \int \phi(|f|) \leqq 1\}$. It is somewhat laborious but not very difficult to show that the semi-inner-product for X is given by:

$$[f, g] = C(g) \int_\Omega f\phi'\left(\frac{|g|}{\|g\|}\right) \operatorname{sgn} g$$

where

$$\operatorname{sgn} g = \begin{cases} \dfrac{|g|}{g} & \text{if } g \neq 0, \\ 0 & \text{if } g = 0 \end{cases}$$

with $C(g) = (\int g\phi'(|g|/\|g\|)\operatorname{sgn} g)^{-1}\|g\|^2$, when g is such that the measure of $\{\xi \in \Omega: \phi$ has no derivative at the point $|g(\xi)|/\|g\|\}$ is 0.

A bounded hermitian operator (see [5]) satisfies by definition $[Hf, f] =$ real for all $f \in X$.

PROPOSITION 1. *If h is real valued and in $L_\infty(\Omega)$, $Hf = hf$ defines a hermitian operator on X, and $\|H\| = \|h\|_\infty$.*

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[1] Actually the proof sketched below covers the Orlicz spaces over measure spaces containing no atoms. If the measure space contains atoms, further argument is needed.