NOTE ON THE COBORDISM RING

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Cobordism is an equivalence relation among compact differentiable manifolds which can be roughly described by: two manifolds are cobordant if together they form the boundary of a differentiable manifold with boundary. Disjoint union and topological product of manifolds induce addition and multiplication operators with respect to which the equivalence classes form a ring graded by dimension. In fact we obtain two rings, depending whether we consider oriented or nonoriented manifolds, and denote them by Ω and $\mathfrak{N}$ respectively. (For precise definitions see [5].)

Thom has shown that $\mathfrak{N}$ is a ring of polynomials mod 2, with one generator x_i in each dimension not of the form 2^i-1, that two nonoriented manifolds are cobordant if and only if they have the same Stiefel numbers, and that x_{2n} can be chosen as the class of the real projective space $P_{2n}(R)$. In [1] Dold gave orientable manifolds V_{2n-1} which can be taken as the odd-dimensional generators.

Much work has been done on the determination of Ω: it has been known for some time that Pontrjagin numbers are invariants of cobordism class, and Thom has shown that $\Omega \otimes Q$ is a polynomial ring, that complex projective spaces $P_{2n}(C)$ can be taken as its generators, that two oriented manifolds determine the same class in it if and only if they have the same Pontrjagin numbers, and that the cobordism groups (graded components of Ω) are all finitely generated Abelian groups. Rohlin studied in [3] the natural map $r: \Omega \to \mathfrak{N}$ obtained by ignoring orientation, and Milnor in [2] showed that Ω has no odd torsion, and that the torsion-free part is a polynomial algebra. These results may be completed by the following, which now determine the algebraic structure of Ω (and incidentally contradict a statement in [4] on which the main theorems of that paper seem to depend):

(1) The cobordism ring Ω contains no elements of order 4.

(2) Two oriented manifolds are cobordant if and only if all their Pontrjagin and Stiefel numbers are the same.

(3) There is a polynomial subalgebra $\mathfrak{W}$ of $\mathfrak{N}$ containing $r(\Omega)$, and a map $\partial: \mathfrak{W} \to \Omega$ such that the following triangle is exact:

$$\begin{tikzcd} \Omega \arrow[rr, "2"] \arrow[dr, "\partial"'] & & \Omega \arrow[dl, "r"] \\ & \mathfrak{w} & \end{tikzcd}$$