DECOMPOSITIONS OF A LOOP INTO CHARACTERISTIC FREE SUMMANDS

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It is known[1] that if the group G is the free sum of proper subgroups G_1 and G_2 (or free product for multiplicative groups), then neither G_1 nor G_2 is normal in G. Clearly, then, there is no nontrivial decomposition of a group into *characteristic* free summands. In this paper we show that the above statement on normality holds for free summands of a loop, but that it is possible, nevertheless, with relatively few restrictions, to obtain decompositions of loops into characteristic free summands.[2]

We use definitions and theorems from a previous paper,[3] referred to throughout as [I], but the statements of these are here repeated for the convenience of the reader.

Notations. If J, K are subsets of a loop L, we denote by $J\cup K$ the set-theoretical sum of elements in J and K; by $J\cap K$ the cross-cut of J and K; and by $J+K$ the subloop of L which is generated by $J\cup K$. If α is a single-valued map of the subset J of L into some loop T, we denote by $J\alpha$ the image of J under α, and by $x\alpha$, the image of any element x in J.

A ***half-loop*** J is a set of elements with a rule of combination, $+$, defined for some ordered pairs of elements (a, b) of J, and subject to the following three conditions:

i. If $a+b=c$ and $a+b=d$, then $c=d$.

ii. If $x+a=y+a$, then $x=y$; if $b+x=b+y$, then $x=y$.

iii. There exists an element, 0, in J, such that $0+x=x+0=x$, for every x in J.

A ***loop*** L is a half-loop satisfying the added condition: If any two of the three elements a, b, c, are in L, then $a+b=c$ determines the third as an element of L. Every subset of a loop L which contains the 0-element of L is a half-loop and, since every half-loop is embeddable in a loop,[4] it follows that every half-loop is a subset of some loop.

By a ***homomorphism*** β of a half-loop J, we mean a single-valued

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Presented to the Society, April 26, 1947; received by the editors August 18, 1947.

[1] See Baer and Levi [1] and Kurosch [1]. The numbers in brackets refer to the bibliography at the end of this paper.

[2] Bruck pointed out the fact that a characteristic subloop need not be normal. See Bruck [1].

[3] See Bates [1].

[4] See Bates [1].