DERIVATIVES OF INFINITE ORDER

R. P. BOAS, JR. AND K. CHANDRASEKHARAN

Let $f(x)$ have derivatives of all orders in (a, b). If, as $n \to \infty$, $f^{(n)}(x) \to g(x)$ uniformly, or even boundedly, dominatedly or in the mean, then $g(x)$ is necessarily of the form ke^x, where k is a constant; in fact, if $c \in (a, b)$,

$$f^{(n-1)}(x) - f^{(n-1)}(c) \to \int_c^x g(t)dt$$

and so

$$g(x) - g(c) = \int_c^x g(t)dt.$$

It then follows first that $g(x)$ is continuous, then that $g(x)$ is differentiable in (a, b), finally that $g'(x) = g(x)$ and so $g(x) = ae^x$.

If $f^{(n)}(x)$ approaches a limit only for one value of x, however, it does not necessarily do so for other values of x. On the other hand, G. Vitali [10][1] and V. Ganapathy Iyer [6] showed that if $f(x)$ is analytic in (a, b) and $f^{(n)}(x)$ approaches a limit for one $x_0 \in (a, b)$, then $f^{(n)}(x)$ converges uniformly in each closed subinterval of (a, b). Ganapathy Iyer asked two questions in this connection:

(I) If $f^{(n)}(x) \to g(x)$ for each x in (a, b), where $g(x)$ is finite, does $g(x) = ke^x$?

(II) If $f(x)$ belongs to a quasianalytic class in (a, b) and $\lim_{n\to\infty} f^{(n)}(x_0)$ exists for a single x_0, does $\lim_{n\to\infty} f^{(n)}(x)$ exist for every x in (a, b)?

We shall show that the answer to both questions is yes. We also indicate some possible generalizations.

We first answer (I).

THEOREM 1. *If $f^{(n)}(x) \to g(x)$ for each x in (a, b), where $g(x)$ is finite, then $f(x)$ is analytic in (a, b).*

It follows from Ganapathy Iyer's result that then $g(x) = ke^x$.

PROOF. At each point x of (a, b) form the Taylor series of $f(x)$. The radius of convergence of this series, as a function of x, has a positive

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[1] Numbers in brackets refer to the references cited at the end of the paper.