LATTICES OF CONTINUOUS FUNCTIONS

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1. **Introduction.** Let X be a compact ($=$bicompact) Hausdorff space and $C(X)$ the set of real continuous functions on X. By defining addition and multiplication pointwise, we convert $C(X)$ into a ring. With the norm $\|f\| = \sup |f(x)|$, $C(X)$ becomes a Banach space. Finally, we may introduce an ordering by defining $f \geqq g$ to mean $f(x) \geqq g(x)$ for all x; this makes $C(X)$ a lattice.

Gelfand and Kolmogoroff [**6**][1] showed that, as a ring alone, $C(X)$ characterizes X. More precisely, if $C(X)$ and $C(Y)$ are isomorphic rings, then X and Y are homeomorphic. Banach [**3**, p. 170] proved that $C(X)$ as a Banach space characterizes X, if X is compact metric. Stone [**5**, p. 469] generalized this to any compact Hausdorff space, and Eilenberg [**5**] and Arens and Kelley [**2**] have since given other proofs. Finally, Stone [**9**] has shown that as a lattice-ordered group, $C(X)$ characterizes X. A negative result is that $C(X)$ as a topological linear space fails to characterize X [**3**, p. 184].

In this paper we shall prove the following result: as a *lattice alone* $C(X)$ characterizes X. This theorem is shown in §5 to subsume all the earlier results cited above. Moreover in this context we can replace the reals by an arbitrary chain, granted a suitable separation axiom. In §4 it is shown that the connectedness of X is equivalent to the indecomposability of $C(X)$ as a lattice.

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2. **Main theorem.** Let R be a chain (simply ordered set). Until §6 it will be assumed that R has neither a minimal nor maximal element. There is a natural way of topologizing R [**4**, p. 27] which can be described as follows: for any $\alpha \in R$ let $U(\alpha)$ be the set of all $\beta \in R$ with $\beta > \alpha$, $L(\alpha)$ the set of all β with $\beta < \alpha$; then the U's and L's form a subbase of the open sets.

LEMMA 1. *If $\alpha, \beta \in R$ and $\alpha > \beta$, then there exist neighborhoods M, N of α, β such that $\gamma > \delta$ for all $\gamma \in M$, $\delta \in N$.*

PROOF. If there exists ξ with $\alpha > \xi > \beta$ we take $M = U(\xi)$, $N = L(\xi)$. If not, we take $M = U(\beta)$, $N = L(\alpha)$.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.