# A NOTE ON FINITE ABELIAN GROUPS[1]

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**1. Introduction.** R. H. Bruck[2] has pointed out that every finite group of odd order is isotopic to an idempotent quasigroup. It can be shown that a necessary and sufficient condition that a group $G$ be isotopic to an idempotent quasigroup is that there exist one-to-one mappings $\theta$ and $\eta$ of $G$ upon $G$ satisfying the relationship $\eta(x)=x\cdot\theta(x)$, for all $x$ of $G$. The same condition is sufficient to prove the existence of a loop $M$ whose automorphism group contains $G$ as a subgroup. We shall not attempt to show either of these applications; but, since there may be others, the present paper is concerned with the existence of suitable $\theta$ and $\eta$ for any finite abelian group $G$. For this we have a complete answer. Our methods are constructive, but (unfortunately from the standpoint of generalization) they make considerable use of the commutative law.

**2. Notation.** We shall consider a finite abelian group $G$ of order $n=n(G)$.

The product of the $n$ distinct elements of $G$ will be designated by $p=p(G)$.

Let $x\to\theta(x)$ be any one-to-one mapping (*not* necessarily an automorphism) of $G$ upon $G$. Consider the derived mapping $x\to\eta(x)=x\theta(x)$. The *order* of $\eta$, denoted by $O(\eta)$, is the number of distinct elements $\eta(x)$, for $x$ in $G$.

It is our purpose to prove the following theorem:

THEOREM 1. *There exists a $\theta$ for which $O(\eta)=n(G)$ unless $G$ possesses exactly one element of order 2. In the latter case there exists a $\theta$ for which $O(\eta)=n(G)-1$.*

**3. Evaluation of $p$.**

LEMMA 1. *$p(G)=1$ unless $G$ possesses exactly one element of order 2. In the latter case, $p(G)$ is the unique element of order 2.*

PROOF. The set $H$ consisting of the identity and all elements of $G$ of order 2 is a uniquely defined subgroup of $G$. If $a\in G$ is of order

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[2] R. H. Bruck, *Some results in the theory of quasigroups*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 19–52, especially pp. 35, 36.