A REMARK ON LOCALLY COMPACT ABELIAN GROUPS

GEORGE W. MACKEY

It has recently been shown by Halmos [1][1] that there exists a compact topological group which is algebraically isomorphic to the additive group of the real line, an example being given by the character group of the discrete additive group of the rationals. Exploiting his argument a bit further it is easy to see that the most general such example is the direct sum of $\aleph$ replicas of the one already given where $\aleph$ is a cardinal such that $2^{\aleph} \leqq C$. This having been observed it naturally occurs to one to ask for the most general locally compact topological group with the algebraic structure in question. It is the purpose of the present note to give a complete answer to this question. We shall do so by giving a proof of the following theorem.

THEOREM. *Let G be a locally compact topological group which is algebraically isomorphic to the additive group of a linear space over the rationals. Then G is isomorphic*[2] *to a direct sum of four groups G_1, G_2, G_3, and G_4 which may be described as follows. G_1 is the additive group of an n-dimensional ($n=0, 1, 2, \cdots$) real linear space with the customary Euclidean topology; that is, an n-dimensional vector group. G_2 is the direct sum of $\aleph$ replicas of the character group of the discrete additive group of the rationals. G_3 is a discrete group algebraically isomorphic to an $\aleph^-$-dimensional linear space over the rationals. G_4 is algebraically isomorphic to the additive group of a linear space over the rationals and contains a compact subgroup H whose quotient group is a discrete torsion group. H is a direct sum of finitely or infinitely many groups each of which is isomorphic to the additive topological group of the p-adic integers for some prime p. The groups G_1, G_2, G_3, and G_4 are unique up to an isomorphism. Furthermore (modulo isomorphisms) the groups G_4 and H determine one another. Thus the numbers n, $\aleph$, $\aleph^-$, and the function f from the primes to the cardinal numbers which gives the multiplicity of occurrence of each p-adic group in H form a complete set of invariants for the topological group G. There exists a G having any assigned set of invariants. Finally G is algebraically isomorphic to the additive group of the real numbers if and only if it has continuum many elements; that is, if and only if n, $\aleph$, $\aleph^-$ and f are chosen so that no G_i has more*

Received by the editors June 13, 1946.

[1] Numbers in brackers refer to the bibliography at the end of the paper.

[2] In this paper the word isomorphic without qualification means isomorphic as a topological group.