ON WEAKLY ORDERED SYSTEMS

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1. Introduction. By a *weakly ordered system* we mean a system D of elements $x, y, \cdots$ with a binary relation $\succ$ such that

$$(1) \qquad x \succ y \text{ implies } x \neq y$$

and

$$(2) \qquad x \succ y \text{ implies } y \succ x \text{ is false.}$$

The statement "$x \succ y$" may be read "x dominates y." Transitivity is not assumed; a transitive weakly ordered system is a partially ordered system. By a *solution* of a weakly ordered system is meant a set V of elements of D such that (a) $x \in V$ and $y \in V$ implies $x \prec y$ is false and (b) $x \in D - V$ implies $y \succ x$ for some $y \in V$. The concept of solution was introduced in J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton, 1944, where it is proved that a weakly ordered system which is strictly acyclic[1] possesses a solution which is unique, and for which a construction is given. This result suggests the problem of finding conditions for the existence and uniqueness of solutions of weakly ordered systems in general. The simplest examples show that if cycles exist neither the existence nor the uniqueness of solutions can be expected in all cases. For example, the system of three elements $a \succ b \succ c \succ a$ has no solution, while the system of four elements $a \succ b \succ c \succ d \succ a$ has the two solutions (a, c) and (b, d). The purpose of this note is to prove the existence of solutions for certain non-acyclic systems. The proof will itself provide a method of construction for the solutions. Zermelo's axiom of choice, the well-ordering theorem, and transfinite induction will be used. The result presented below is a contribution to the general problem suggested above rather than to the theory of games. For the hypothesis of the theorem below precludes transitivity completely; that is, it precludes the existence of three elements a, b, c, such that $a \succ b$, $b \succ c$, and $a \succ c$. This restriction is too severe for the theory of games, just as is the assumption of transitivity. The problem remains open for weakly ordered systems which are not strictly acyclic but also do not satisfy the hypothesis of the theorem below.

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[1] See loc. cit. pp. 590–600 for definitions and proof.