SOME REMARKS ON ALMOST PERIODIC TRANSFORMATIONS

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In a recent paper in this Bulletin (see [3][1]), W. H. Gottschalk has proved a number of interesting theorems on "recurrent" and "almost periodic" homeomorphisms of a space on itself. In the first part of the present note we give very simple proofs of some of Gottschalk's theorems in an even more general form. In the second half we consider "regular" transformations in more detail.

1. Recurrent and almost periodic transformations.

Notations. Let f be a continuous mapping (not necessarily a homeomorphism) of a topological space X in itself (that is, $f(X) \subset X$). We say that f is *recurrent* at a point $x \in X$, or that x is *recurrent* under f, if, given any neighbourhood $U(x)$ of x, there exist infinitely many positive integers n for which $f^n(x) \in U(x)$. (This definition is equivalent to Gottschalk's if X is a T_1 space.) Further, f is *almost periodic* at x if, given any $U(x)$, there exists an $N(x, U(x)) > 0$ such that for the (infinite) sequence $\{n_i\}$ of positive integers for which $f^{n_i}(x) \in U(x)$ we have $n_{i+1} - n_i \leqq N$.

THEOREM I. *If a continuous mapping f of a topological space X in itself is either* (a) *recurrent, or* (b) *almost periodic, at x, then so is f^k, for each positive integer k.*[2]

PROOF. Let N_r denote the class of positive integers congruent to r mod k. We may clearly assume that one at least of the classes $N_1, N_2, \cdots, N_{k-1}$, say N_r, satisfies: every neighbourhood $U(x)$ of x contains $f^n(x)$ for infinitely many values of $n \in N_r$; for otherwise each $U(x)$ will contain $f^n(x)$ for all large enough $n \in N_k$, and the theorem will follow trivially.

Now let U_0 be any given open set containing x. Choose $n_1 \in N_r$ such that $f^{n_1}(x) \in U_0$. Since f^{n_1} is continuous, there exists an open set $U_1 \ni x$ such that $U_1 \subset U_0$ and $f^{n_1}(U_1) \subset U_0$. Choose $n_2 \in N_r$ such that $f^2(x) \in U_1$; and so on. In this way, we define integers $n_1, \cdots, n_{k-1} \in N_r$ and open sets $U_1 \supset U_2 \supset \cdots \supset U_{k-1} \ni x$ such that $f^{n_i}(x) \in U_{i-1}$ and $f^{n_i}(U_i) \subset U_{i-1}$.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Theorem I(a) is Theorem 1 of [3], without the restriction that f be a homeomorphism. Theorem I(b) is Theorem 6 of [3], without the restrictions that f be a homeomorphism and that X be compact.