DISTRIBUTIVE PROPERTIES OF SET OPERATORS

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A set operator, designated by small letters, $a, b, \cdots$, is one which takes subsets $A, B, \cdots$ of a given space S into subsets $aA, aB, \cdots \subset S$. A *property* of a set operator a is a constant relation between argument and image sets under a and is expressed in a statement of equation or inclusion, for example, $a(A+B)=aA\cdot aB$ or $A\subset bA$ where a, b may mean, for example, "complement" or "closure."

We investigate properties expressed by relations of the form:

$$a(Af_1B)\cdot\mathcal{R}\colon aAf_2aB \tag{1}$$

where f_1, f_2 are either set sum: $+$, or set product: $\cdot$, and where $\mathcal{R}$ is either $=$, $\supset$, or $\subset$. A property defined by such a relation (1) is a distributive property, but not all distributive properties can be defined by (1), for example $a(A+B)=A\cdot aB+B\cdot aA$, and so on.

When f_1, f_2, $\mathcal{R}$ are given constant values, (1) becomes the statement of a specific distributive property of a. We now list them individually for reference. Properties of *monotonicity* and *inverse-monotonicity* (α_{13} and α_{14} below) are closely related to properties of distributivity so they are listed in the table also. (The arrow, $\rightarrow$, is used for implication throughout this paper.)

TABLE I

α_1:	$a(A+B)=aA+aB$	α_7:	$a(A+B)=aA\cdot aB$
α_2:	$a(A+B)\supset aA+aB$	α_8:	$a(A+B)\supset aA\cdot aB$
α_3:	$a(A+B)\subset aA+aB$	α_9:	$a(A+B)\subset aA\cdot aB$
α_4:	$a(A\cdot B)=aA\cdot aB$	α_{10}:	$a(A\cdot B)=aA+aB$
α_5:	$a(A\cdot B)\supset aA\cdot aB$	α_{11}:	$a(A\cdot B)\supset aA+aB$
α_6:	$a(A\cdot B)\subset aA\cdot aB$	α_{12}:	$a(A\cdot B)\subset aA+aB$
α_{13}:	$A\subset B\rightarrow aA\subset aB$		
α_{14}:	$A\subset B\rightarrow aA\supset aB$		

To say that a has property α_1 (notation $a\colon\alpha_1$) means: "For every A, B, $a(A+B)=aA+aB$." These properties α_i are obviously not independent, for example, $a\colon\alpha_1\rightarrow a\colon\alpha_2, \alpha_3$ (which we may shorten, at our convenience, to $\alpha_1\rightarrow\alpha_2, \alpha_3$).

Our first main question is: if we hypothesize to a a single property α_i, what other properties *must* a have? This is completely answered by the following diagram of implications:

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