GROUPS WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS

REINHOLD BAER

If f is a homomorphism of the group G, and if g is an isomorphism of the image group G^f, then fg is a homomorphism of G too; and this homomorphism is an isomorphism if, and only if, f is an isomorphism. Consequently the following three properties of the group G imply each other.

(1) *Homomorphisms of G upon isomorphic groups are isomorphisms.*

(2) *Homomorphisms of G upon itself are isomorphisms.*

(3) *If N is a normal subgroup of G such that G and G/N are isomorphic, then $N=1$.*

Groups meeting these requirements (1) to (3) shall be termed Q-groups. A direct product of an infinity of isomorphic groups different from 1 is certainly not a Q-group. On the other hand it is readily seen that groups satisfying the ascending chain condition for normal subgroups are Q-groups. Deeper is the fact that the group G is a Q-group if it belongs to one of the following three classes of groups.

(a) Free groups on a finite number of generators.[1]

(b) Groups of finite dimensional matrices with coefficients from a field, which are generated by a finite number of elements.[2]

(c) Free products of a finite number of abelian groups each of which is generated by a finite number of elements.[3]

All these groups are generated by a finite number of elements. But we are able to show that this latter condition is neither necessary nor sufficient for a group to be a Q-group. The complexity of the situation is increased by the fact that neither subgroups nor quotient groups of Q-groups need be Q-groups. Thus it becomes desirable to obtain general criteria for a group to be a Q-group, and this is the main object of the present note.

The subgroup S of the group G shall be termed *completely characteristic, if $S=S^f$ for every homomorphism f of G which satisfies $G=G^f$.* Examples of completely characteristic subgroups are the commutator subgroup of G and its generalizations. If G happens to be a Q-group, then every characteristic subgroup of G is completely characteristic.

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[1] Levi [1]; Magnus [2]. The numbers in brackets refer to the bibliography at the end of the paper.

[2] Malcew [1].

[3] Fouxe-Rabinowitch [1].