ON THE SIEVE METHOD OF VIGGO BRUN

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1. **Introduction.** Numerous improvements have been made in the the sieve method since the appearance of Brun's work.[1] Rademacher,[2] Estermann,[3] and, more recently, Buchstab[4] have introduced new ideas and so obtained more precise results. If their work is examined it will be seen that the various estimates which they use can all be made to depend on the single estimate

$$\sum_{p \leqq x} (\log p)/p = \log x + O(1), \tag{1.1}$$

the summation extending over all primes $p \leqq x$.

In a previous paper[5] it was shown that Rademacher's results could be extended to any infinite set of primes for which the estimate

$$\sum_{p \leqq x}{}' (\log p)/p = r \log x + O(1) \tag{1.2}$$

applies. Here, and subsequently, the dash indicates that the summation extends over all the primes of a given infinite set which do not exceed x, and r is a given real number such that $0 < r \leqq 1$.

In this paper it will be shown that Buchstab's results can be similarly extended. Application will be made to primes in arithmetic progression since the set of all primes $p \equiv h \pmod{k}$ with $(h, k) = 1$ satisfies (1.2) with $r = 1/\phi(k)$.

2. **Preliminary lemmas.** It is well known[6] that from (1.1) it follows that

$$\sum_{p \leqq x} 1/p = \log \log x + C + O(1/\log x)$$

and

$$\prod_{p \leqq x} (1 - (2/p)) = D/\log^2 x + O(1/\log^3 x),$$

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[1] *Le crible d'Eratosthene et le théorème de Goldbach*, Christiania, 1920

[2] Abh. Math. Sem. Hamburgischen Univ. vol. 3 (1924) pp. 12–30.

[3] J. Reine Angew. Math. vol. 168 (1932) pp. 106–116.

[4] Rec. Math. (Mat. Sbornik) N.S. vol. 46 (1938) pp. 375–387. C. R. (Doklady) Acad. Sci. URSS. vol. 29 (1940) pp. 544–548.

[5] Trans. Amer. Math. Soc. vol. 43 (1938) pp. 296–302.

[6] Landau, *Primzahlen*, vol. I, pp. 98–102.