CONCERNING SETS OF POLYNOMIALS ORTHOGONAL SIMULTANEOUSLY ON SEVERAL CIRCLES

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Introduction. Sets of polynomials $\{p_n(z)\}$ simultaneously orthogonal on several curves have been investigated by J. L. Walsh and the author. Recently, the particular case of circles has been treated by G. M. Merriman,* and it is the purpose of this note to give a shorter derivation of his result.

For $r \geqq 0$, $m \geqq 1$ the polynomials

$$1, z, z^2, \cdots, z^{m-1}; \quad z^{n-m}(z^m - r^m), \qquad n = m, m+1, \cdots, \tag{1}$$

are orthogonal on the circles $|z| = R > r$ with the weight function

$$w(z) = |z^m - r^m|^{-2}. \tag{2}$$

From the work of Walsh and the author† we know that if a set of polynomials is orthogonal on two distinct circles, these circles must be concentric, and a very simple relation holds between the corresponding weight functions. According to the results of the author, the case (1), (2) is the only one, save for integral linear transformations, in which a set of polynomials is simultaneously orthogonal on all circles concentric to a given circle and containing it. Merriman shows that this holds true if only the simultaneous orthogonality on two (necessarily concentric) circles is assumed.

Preliminary remarks. In order to prove this pretty result, we shall use the following simple identity satisfied by the polynomials

$$p_n(z) = k_n z^n + \cdots, \qquad k_n > 0; n = 0, 1, 2, \cdots, \tag{3}$$

which constitute an orthonormal set on the unit circle $|z| = 1$ with a preassigned weight function $w(t) = w(e^{i\theta}) = f(\theta)$:

$$\sum_{\nu=0}^{n} \overline{p_\nu(0)} p_\nu(z) = k_n z^n \bar{p}_n(z^{-1}). \tag{4}$$

The reader may find this identity in my earlier paper, *Beiträge zur Theorie der Toeplitzschen Formen*, II (Mathematische Zeitschrift, vol. 9 (1921), pp. 167–190, in particular, p. 174, (33)). For the sake of completeness, however, I include here a very simple direct proof for it.

* See a note of the same title as the present one in this Bulletin, vol. 44 (1938), pp. 57–69. Here also references can be found to the literature on the subject.

† Cf. loc. cit.