A NEW DEFINITION OF A STIELTJES INTEGRAL*

BY A. H. COPELAND

Let $f(x)$ be a function of limited variation defined on (α, β), and $x_1, x_2, \cdots, x_n$ a sequence which divides (α, β) into a finite number of intervals. Let $g(x)$ be bounded on (α, β). If

$$\text{(A)} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\{f(x_i) - f(x_{i-1})\}$$

exists when $x_i - x_{i-1}$ tends to zero, this limit is the Riemann-Stieltjes integral of g with respect to f. If g is continuous the integral exists. If g is not continuous (A) need not exist. In particular (A) does not exist when g is of bounded variation unless restrictions which are conditioned by g are placed on the sequence $x_1, x_2, \cdots$.

In this note we define an integral of g with respect to f as the Cesàro mean of g formed for a sequence $x_1, x_2, \cdots$ of ordinates. This sequence is defined in terms of f only. When the function f, and consequently the method of formation of the sequence $x_1, x_2, \cdots$, has been given, the integral exists for every g which has discontinuities of the first kind only, and reduces to (A) when g is continuous. For the present we shall assume that f is monotone, $f(\alpha+0)=0$, and $f(\beta)=1$. At a later point we shall show how these restrictions can be removed.

The sequence $x_1, x_2, \cdots$ is related to the function f in the following manner: Let $a < x \leqq b$ be any sub-interval of $\alpha < x \leqq \beta$. Let H_n be the number of points of the sequence $x_1, x_2, \cdots, x_n$

---

* Presented to the Society, December 31, 1930. For other extensions of the concept of the Stieltjes integral, see H. L. Smith, *On the existence of the Stieltjes integral*, Transactions of this Society, vol. 27 (1925), p. 491; S. Pollard, *The Stieltjes integral and its generalizations*, Quarterly Journal of Mathematics, vol. 49 (1920–3), p. 73; Kolmogoroff, *Untersuchungen über den Integralbegriff*, Mathematische Annalen, vol. 103 (1930), p. 654; Young, *The algebra of many-valued quantities*, Mathematische Annalen, vol. 104 (1931), p. 260, and *Many-valued Riemann-Stieltjes integration*, Cambridge Philosophical Society Proceedings, vol. 27 (1931), p. 325; Ben Dushnik, *On the Stieltjes integral*, published by Edwards Brothers (Ann Arbor).