MATRICES WITH ELEMENTS IN A PRINCIPAL IDEAL RING*

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1. *Rings.* To attempt to distinguish between algebra and number theory is probably futile, but, speaking approximately, it may be said that algebra (in the narrowest sense of the word) is the study of fields, while number theory is the study of rings. The mathematical system which seems most satisfactory as an abstraction of the system of rational integers is the principal ideal ring. By this I mean that the basic theorems of number theory, such as unique factorization into primes, hold for a principal ideal ring, while the concept of principal ideal ring is sufficiently general to include many other instances besides the rational integers.

A *ring*† is a mathematical system composed of more than one element, an equals relation, and two operations, $+$ and $\times$, subject to the following laws. The elements form an abelian group relative to the operation $+$, the identity element being denoted by 0. The set of elements is closed under the operation $\times$, which is associative. Finally, the operation $\times$ is distributive with respect to the operation $+$.

If $a\neq 0$ and $b\neq 0$ are elements of a ring $\mathfrak{R}$ such that $ab=0$, then a and b are called *divisors of zero*. A commutative ring without divisors of zero is called a *domain of integrity*.

Let a, b, c be elements of a domain of integrity $\mathfrak{D}$. If $ab=c$, then $a\,|\,c$ (a divides c), $b\,|\,c$, and a and b are called *divisors* of c. If $a\,|\,b$ and $a\,|\,c$, then a is called a *common divisor* of b and c. If, furthermore, every common divisor of b and c divides a, then a is a *greatest common divisor* (g. c. d.) of b and c.

If there exists a number 1 of $\mathfrak{D}$ such that $1\cdot a=a\cdot 1=a$ for every a, this number 1 is called a *principal unit*.‡ A domain of integrity with a principal unit in which every pair of elements

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† In the interest of uniformity I have used the definitions of van der Waerden, *Moderne Algebra*, Springer, 1930–31.

‡ *Einselement*, van der Waerden.