THE NORM OF A SPACE CONFIGURATION

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1. *Introduction.* In a recent paper,* the writer has shown that attached to an ordered set of $2n$ points in a plane there is the lineo-linear invariant

$$2(N + iA) = \sum_{k=1}^{n} \bar{x}_{2k}(x_{2k-1} - x_{2k+1}) = \left(\frac{1}{n}\right) \sum_{i=1}^{n-1} (1 - \epsilon^{n-i}) v_i \bar{u}_i,$$

where x_k $(k = 1, 2, \cdots, 2n)$ are points in the complex plane, $\bar{x}_k$ denotes the conjugate of x_k, ϵ is a primitive nth root of unity, A is the area of the ordered $2n$-gon, and the *norm* N is defined by

$$2N = -\tfrac{1}{2}(\delta_{12}^2 - \delta_{23}^2 + \cdots + \delta_{2n-1,2n}^2 - \delta_{2n,1}^2),$$

with $\delta_{ij} = |x_i - x_j|$. The Lagrange resolvents†

$$v_i = \sum_{k=0}^{n-1} \epsilon^{ik} x_{2k+1}, \quad \bar{u}_i = \sum_{k=0}^{n-1} \epsilon^{i(n-k)} \bar{x}_{2(k+1)},$$
$$(i = 1, 2, \cdots, n-1),$$

are absolute invariants under translations $y_i = x_i + b$, while the combinations $v_i \bar{u}_i$ are likewise invariant under rotations $y = tx_i$, where t is a complex number with unit modulus.

This paper extends the preceding results to S_3, the theorems that are obtained holding, mutatis mutandis, for S_n. Denoting the $2n$ points by $X_i (i = 1, 2, \cdots, 2n)$ and calling the two sets X_{2i}, X_{2i-1} $(i = 1, 2, \cdots, n)$ the *component* n-points of the whole set, it is shown that the norm of the $2n$-point is expressible in terms of the Lagrange resolvents of its vertices and is *consequently absolutely invariant under a translation of either of its component n-points.*

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* *Lagrange resolvents in eucilidean geometry,* American Journal of Mathematics, vol. 49 (1927). pp. 511–522.

† Pascal, E., Repertorium, 2d. ed., vol. 1, p. 307.