NOTE ON THE THEOREM OF GENERALIZED FOURIER'S CONSTANTS.

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In the theory of the development of arbitrary functions $f(x)$ in series of normalized orthogonal functions $\psi_i(x)$,

$$f(x) = \sum_1^\infty a_i\psi_i(x),$$

$$a_i = \int_a^b f(x)\psi_i(x)dx, \qquad \int_a^b \psi_i(x)\psi_k(x)dx \begin{cases} = 0, & i \neq k, \\ = 1, & i = k, \end{cases}$$

sufficient conditions that this equality exists, and that the series converges uniformly, are in general that $f(x)$ and its first $m-1$ derivatives are continuous in (a, b) and satisfy homogeneous boundary conditions for $x = a$ and $x = b$.* Then there follows immediately the fundamental theorem of "generalized" Fourier's constants

$$\int_a^b \{f(x)\}^2 dx = \sum_1^\infty a_i^2. \tag{1}$$

This note will give a simple proof that, in case (1) holds true for every function satisfying the above conditions, it holds true for every integrable function $f(x)$, such that $\{f(x)\}^2$ is integrable.†

Since $f(x)$ and $\{f(x)\}^2$ are integrable in (a, b) there exists, for every $\epsilon > 0$, a division of (a, b) in a finite number of subintervals $(x_1, x_2), (x_2, x_3), \cdots, (x_{n-1}, x_n)$, $x_1 = a$, $x_n = b$, such that a function $\phi(x)$ can be defined, having the following properties:

$$\begin{gathered} |\phi(x)| = \text{lower bound of } f(x) \text{ in } (x_i, x_{i+1}) \text{ for } x_i \leqq x < x_{i+1}, \\ \phi(x)f(x) \geqq 0, \\ \left| \int_a^b [\{f(x)\}^2 - \{\phi(x)\}^2] dx \right| < \frac{\epsilon}{2}. \end{gathered} \tag{2}$$

* D. Hilbert, "Zweite Mitteilung über Integralgleichungen," *Göttinger Nachrichten*, 1904.
E. Schmidt, Dissertation, Göttingen, 1905.

† The theorem has been proven essentially by W. Stekloff with the restriction that $f(x)$ be bounded, *Mémoires de l'Académie de St. Pétersbourg*, 1904. The above proof is simpler and does away with this restriction.