BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, March 1972

INVARIANT SUBSPACE THEORY FOR THREE-DIMENSIONAL NILMANIFOLDS

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Communicated by C. C. Moore, September 15, 1971

1. **Introduction.** Let N denote the nilpotent Lie group whose underlying manifold is three-dimensional Euclidean space $\boldsymbol{R}^3$ and whose group operation is given by $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$. The subset $\Gamma = \{(a, b, c): a, b, c \in \boldsymbol{Z}\}$ of N is a subgroup, and the quotient N/Γ is a compact manifold, denoted $\boldsymbol{M}$. On the manifold $\boldsymbol{M}$ there is a unique probability measure ν invariant under translation by N. (We use right cosets Γg, $g \in N$, and hence translation here means right-translation.) We will use R to denote the regular representation of N on $L^2(\boldsymbol{M}, \nu)$, namely: $(R_g\phi)(\Gamma h) = \phi(\Gamma hg)$ for all $g, h \in N$ and all $\phi \in L^2(\boldsymbol{M}, \nu)$.

The representation R decomposes into a direct-sum of irreducible subrepresentations. However, some of the irreducible representations in the sum occur with multiplicity greater than 1, and consequently, $L^2(\boldsymbol{M}, \nu)$ does not decompose *uniquely* into a direct sum of irreducible R-invariant subspaces. The theorems announced below are aimed toward remedying this situation by introducing into the family of all irreducible R-invariant subspaces of $L^2(\boldsymbol{M}, \nu)$ a certain amount of structure.

Let $\mathfrak{z}N$ denote the center of N. The Stone-von Neumann theorem says that for each nonzero real number ξ, there is a unique (up to unitary equivalence) irreducible unitary representation U^ξ whose restriction to $\mathfrak{z}N$ is a multiple of the character $(0, 0, z) \mapsto e^{2\pi i\xi z}$ of $\mathfrak{z}N$. We will use $L(\xi)$ to denote the Hilbert space of U^ξ.

It is easy to see that, aside from the characters of N vanishing on Γ, the only irreducible summands of R are those U^ξ where ξ is a nonzero integer. In fact, let n be a nonzero integer, and let $H(n)$ denote the subspace of $L^2(\boldsymbol{M}, \nu)$ consisting of those functions f satisfying $(R_{(0,0,z)}f)(\Gamma h) = e^{2\pi inz}f(\Gamma h)$ for all $h \in N$ and $(0, 0, z) \in \mathfrak{z}N$; then the restriction of R to $H(n)$ is unitarily equivalent to the representation $U^n \otimes 1$ of N on $L(n) \otimes \boldsymbol{C}^{|n|}$. (For a proof, see C. C. Moore [**2**].) It follows that the irreducible subspaces of $H(n)$ are in one-to-one correspondence with the space of lines in $\boldsymbol{C}^{|n|}$ through 0—that is, projective space $\boldsymbol{CP}^{|n|-1}$. The theorems below refine this observation.

AMS 1969 *subject classifications.* Primary 2265.
Key words and phrases. Nilmanifold, harmonic analysis.

[1] John Simon Guggenheim Fellow.
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