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A NEW FIXED POINT THEOREM AND ITS APPLICATION

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Introduction. The purpose of this note is two-fold. In §1 we present a general fixed point theorem (Theorem 1 below) for 1-set-contractions and 1-ball-contractions defined on closures of bounded open subsets of Banach spaces. In §2 we indicate briefly how Theorem 1 is used to deduce a number of known, as well as some new, fixed point theorems for various special classes of mappings which recently have been extensively studied and which are shown to be either 1-set-contractive or 1-ball-contractive. Complete proofs and detailed discussion of the results presented in this note will be given in [**14**].

1. **A fixed point theorem.** Let X be a real Banach space, D a subset of X with $\bar{D}$ denoting its closure and ∂D its boundary, T a bounded continuous mapping of $\bar{D}$ into X. Following Kuratowski we say that T is *k-set-contractive* if $\gamma(T(A)) \leqq k\gamma(A)$ for each bounded $A \subset \bar{D}$ and some constant $k \geqq 0$, where $\gamma(A)$ is the set-measure of noncompactness of A given by

$$\inf\{r > 0 | A \text{ can be covered by a finite number of sets of diameter} \leqq d\}.$$

An important example of a k-set-contraction, $k < 1$, is a mapping $T = S + C$ with $S:\bar{D} \to X$ strictly contractive (i.e., $\|Sx - Sy\| \leqq k\|x - y\|$ for $s, y \in \bar{D}$, $k < 1$) and $C:\bar{D} \to X$ compact. In [**16**] Sadovsky introduced a related class of mappings to which we refer here as *ball-condensing*, i.e., $T:\bar{D} \to X$ is such that $\chi(T(A)) < \chi(A)$ for each bounded $A \subset \bar{D}$, where $\chi(A)$ is the ball-measure of noncompactness of A given by

$$\inf\{r > 0 | A \text{ can be covered by a finite number of balls with centers in } X \text{ and radius } r\}.$$

In analogy with k-set-contractions, we say that $T:\bar{D} \to X$ is *k-ball-contractive* if $\chi(T(A)) \leqq k\chi(A)$ for each bounded $A \subset \bar{D}$ and some $k \geqq 0$. The two classes of mappings, k-set-contractions and k-ball-contractions, are in fact different since they are defined in terms of measures γ and χ which are known to be different although they have a great deal in common. It follows that a k-set-contraction with $k < 1$ is set-condensing and that a set-condensing map is 1-set-contractive, but the reverse implications

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