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FINITE MODULES AND ALGEBRAS OVER DEDEKIND DOMAINS AND ANALYTIC NUMBER THEORY

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This note states some results concerning asymptotic enumeration of the isomorphism classes of finite modules or algebras (of various types) over a Dedekind domain D. Proofs will be published elsewhere.

1. **Finite modules over a ring of algebraic integers.** Firstly, let D be the ring of integers in a finite-dimensional algebraic number field K. If M is a finitely-generated torsion module over D, then standard structure theory [**8**], [**9**] and the fact that D/P is finite for every prime ideal P implies that M is *finite in cardinal*. Further, if $\mathscr{F}(D)$ denotes the category of all such modules M and $a(n) = a_D(n)$ denotes the *total number of isomorphism classes of modules of order n in* $\mathscr{F}(D)$, then $a(n)$ is finite and "multiplicative."

Now recall that, if $N_D(x)$ denotes the total number of ideals of norm at most x in D, then $N_D(x) = \lambda_K x + O(x^\eta)$ where λ_K is an explicit positive constant depending on K and $\eta = 1 - 2/(1 + [K:\boldsymbol{Q}])$ [**13**].

(1.1) THEOREM. *The function* $a(n)$ *has mean value* $\lambda_K \prod_{r=2}^{\infty} \zeta_K(r)$. *More precisely,* $\sum_{n \leqq x} a(n) = [\lambda_K \prod_{r=2}^{\infty} \zeta_K(r)]x + O(x^{1/2})$ *where* $\zeta_K(s)$ *is the Dedekind zeta function.*

When D is the ring $\boldsymbol{Z}$ of *rational integers*, $\mathscr{F}(D)$ becomes the category $\mathscr{A}$ of all ordinary *finite abelian groups*, and the theorem was first proved for this case by Erdös and Szekeres [**4**].

(1.2) COROLLARY. *Let* $\pi_{\mathscr{F}(D)}(x)$ *denote the total number of indecomposable D-modules of order at most* x *in* $\mathscr{F}(D)$. *Then*

$$\pi_{\mathscr{F}(D)}(x) \sim x/\log x \quad \textit{as } x \to \infty.$$

Theorems 1.1 and 2.1 follow from slightly more general results about certain categories. Corollaries 1.2 and 2.2 follow with the aid of an *abstract prime number theorem*, as discussed in [**15**]; for $D = \boldsymbol{Z}$, see [**10**], [**11**].

Although it has a finite mean value, $a(n)$ can be very large on prime powers: Consider a rational prime p, and define $C = C(D, p)$ by $C = \alpha_1^{-1} + \cdots + \alpha_m^{-1}$ where $(p) = P_1 \cdots P_m$ is the decomposition of (p) into prime ideals P_i in D, and P_i has norm p^{α_i}.

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