LACUNARY TAYLOR AND FOURIER SERIES

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To the memory of Jacques Hadamard

Introduction. The history of lacunary Fourier and Taylor series goes back to Weierstrass and Hadamard, if not to Riemann.

According to Weierstrass [**49**], Riemann told his students in 1861 that the continuous function

$$(1) \qquad \sum_{1}^{\infty} \frac{\sin n^2 x}{n^2}$$

is nowhere differentiable. As Weierstrass was not able to prove it (and, in fact, until now, it seems to have been neither proved nor disproved), he gave (1872) his famous example

$$(2) \qquad \sum_{1}^{\infty} a^n \cos \lambda^n x$$

where λ is an odd integer $\geqq 3$, and a a positive number such that $a < 1$ and $a\lambda > 1 + 3\pi/2$: (2) is a continuous function which is nowhere differentiable [**49**]. Later on, Weierstrass's result was improved by Hardy: the previous statement holds under the assumption $a\lambda \geqq 1$ instead of $a\lambda > 1 + 3\pi/2$ [**12**]. In Hardy's version, that is a rather hard theorem; as we shall see later, it can be made very easy.

Hadamard (1892) proved that the Taylor series

$$(3) \qquad \sum_{1}^{\infty} a_n z^{\lambda_n} \qquad \limsup_{n\to\infty} |a_n|^{1/\lambda_n} = 1$$

has $|z| = 1$ as a natural boundary, whenever there exists a $q > 1$ such that

$$(4) \qquad \frac{\lambda_{n+1}}{\lambda_n} > q > 1 \qquad (n = 1, 2, \cdots) \qquad [\mathbf{11}, \text{p. } 116].$$

(4) is known as Hadamard's lacunarity condition. We shall see that Hadamard's condition has played quite an important part in many directions. However, it is not what is needed about $\{\lambda_n\}$ to get that

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