PAIRS OF INVERSE MODULES IN A SKEWFIELD

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Let Σ be a skewfield. If J and J' are submodules of Σ such that the nonzero elements of J are the inverse elements of those of J', then J and J' form a "pair of inverse modules." A module admitting an inverse module will be called a J-module and a selfinverse module containing 1 will be called an S-module. In an earlier paper[1] the author has shown that if Σ is a (commutative) field of characteristic not equal to 2, then every S-module is a subfield of Σ. Only in fields of characteristic 2, nontrivial S-modules can be found. A corresponding distinction of that characteristic does not hold for skewfields. Even the skewfield of the quaternions contains nontrivial S-modules, for examples the module generated by 1, j, k. In the present paper some properties of S-modules and J-modules will be discussed. For example it will be proved that when an S-module contains the elements a, b and ab, it contains all the elements of the skewfield which is generated by a and b. By a similar method it will be shown that finite S-modules are necessarily Galois-fields.

1. Necessary and sufficient conditions for J-modules.

THEOREM 1. *A submodule J of Σ is a J-module if and only if $a \in J$ and $b \neq 0 \in J$ imply $ab^{-1}a \in J$.*

PROOF. Let J be a J-module. Without loss of generality suppose that $a \neq 0$, $b-a=c\neq 0$. Then $k=a^{-1}+c^{-1}\in J'$ since J' is closed under addition and subtraction. As $k=a^{-1}(c+a)c^{-1}$, $k^{-1}=cb^{-1}a$; hence $a-k^{-1}=ab^{-1}a$ is contained in J. Let now J be a module satisfying the condition mentioned above. To prove that J is a J-module, we shall show that when a and c are nonzero elements in J, but otherwise arbitrary, then $a^{-1}+c^{-1}$ is either 0 or the inverse of an element of J. The first alternative holds when $b=a+c=0$; if however $b\neq 0$, then $a^{-1}+c^{-1}=(a-ab^{-1}a)^{-1}$ is the inverse of an element of J. Hence the theorem.

COROLLARY 1. *The meet of any (finite or infinite) set of J-modules in Σ is a J-module in Σ.*

This corollary shows that the J-modules in Σ form a lattice with the set-inclusion as the defining order-relation. $J_1 \wedge J_2$ denotes the ordi-

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[1] *Pairs of inverse moduls*, J. Indian Math. Soc. N.S. vol. 3 (1936) pp. 295–306.