Asian J. Math.
Vol. 14, No. 2, pp. 235–242, June 2010

COMPLEX PRODUCT MANIFOLDS AND BOUNDS OF CURVATURE*

LUEN-FAI TAM† AND CHENGJIE YU‡

Abstract. Let $M = X \times Y$ be the product of two complex manifolds of positive dimension. In this paper, we prove that there is no complete Kähler metric g on M such that: either (i) the holomorphic bisectional curvature of g is bounded by a negative constant and the Ricci curvature is bounded below by $-C(1+r^2)$ where r is the distance from a fixed point; or (ii) g has nonpositive sectional curvature and the holomorphic bisectional curvature is bounded above by $-B(1+r^2)^{-\delta}$ and the Ricci curvature is bounded below by $-A(1+r^2)^{\gamma}$ where A, B, γ, δ are positive constants with $\gamma + 2\delta < 1$. These are generalizations of some previous results, in particular the result of Seshadri and Zheng [8].

Key words. Complex products, Kähler manifolds, bisectional curvature, negative curvature.

AMS subject classifications. Primary 53B25; Secondary 53C40

1. Introduction.

In [8], Seshadri and Zheng proved the following result:

Theorem 1.1. *Let $M = X \times Y$ be the product of two complex manifolds of positive dimension. Then M does not admit any complete Hermitian metric with bounded torsion and holomorphic bisectional curvature bounded between two negative constants.*

In particular, there is no complete Kähler metric on M with holomorphic bisectional curvature bounded between two negative constants. For earlier results in this direction see [11, 15, 16, 7]. The result of Seshadri-Zheng has been generalized by Tosatti [10] to almost-Hermitian manifolds.

On the other hand, there is an open question whether the assumption on the lower bound of the curvature can be removed. In fact, it is an open question raised by N. Mok (see [8]) whether the bidisc admit a complete Kähler metric with holomorphic bisectional curvature bounded from above by -1.

In this work, by using a local version of the generalized Schwartz lemma of Yau [14] and an Omori-Yau type maximum principle of Takegoshi [9], we prove the following:

Theorem 1.2. *Let $M = X^m \times Y^n$ be the product of two complex manifolds of positive dimension. Then, there is no complete Kähler metric on M with Ricci curvature $\geq -A(1+r)^2$ and holomorphic bisectional curvature $\leq -B$, where A and B are some some positive constants, and $r(x,y) = d(o,(x,y))$ is the distance of (x,y) and a fixed point $o \in M$.*

On the other hand, Seshadri [7] has constructed a complete Kähler metric on $\mathbb{C}^n$ with negative curvature. It seems that the assumption on the upper bound of the curvature in Theorem 1.1 is necessary. However, one can also relax the upper bound of the curvature as follows:

*Received September 15, 2009; accepted for publication March 8, 2010.

†The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China (lftam@math.cuhk.edu.hk). Research partially supported by Hong Kong RGC General Research Fund #CUHK 403108.

‡Department of Mathematics, Shantou University, Shantou, Guangdong, China (cjyu@stu.edu.cn). Research partially supported by NSFC (11001161), NSFC (10901072) and GDNSF (9451503101004122).